Available online at www.sciencedirect.com
ScienceDirect
Stochastic Processes and their Applications 119 (2009) 45–73
stochastic processes and their applications
www.elsevier.com/locate/spa

Stochastic coalescence with homogeneous-like interaction rates

Nicolas Fournier, Eva Löcherbach*

Centre de Mathématiques, Faculté de Sciences et Technologie, Université Paris-Est, 61 avenue du Général de Gaulle, 94010 Créteil Cedex, France

Received 17 August 2007; received in revised form 10 January 2008; accepted 16 January 2008
Available online 19 January 2008

Abstract

We study infinite systems of particles characterized by their masses. Each pair of particles with masses x and y coalesces at a given rate $K(x, y)$. We consider, for each $\lambda \in \mathbb{R}$, a class of homogeneous (or homogeneous-like) coagulation kernels K. We show that such processes exist as strong Markov–Feller processes with values in ℓ_λ, the set of ordered $[0, \infty]$-valued sequences $(m_i)_{i\geq 1}$ such that $\sum_{i\geq 1} m_i^\lambda < \infty$.

MSC: 60K35; 60J25

Keywords: Coalescence; Homogeneous coagulation kernels; Stochastic interacting particle systems

1. Introduction

We consider a possibly infinite system of particles characterized by their masses, in which each pair of particles with masses x and y merge into a single particle with mass $x + y$ at some given rate $K(x, y)$, which we will refer to as the *coagulation kernel*. We refer to the review of Aldous [3] on stochastic coalescence, and on its links with the Smoluchowski coagulation equation.

When the initial state consists of a finite number of particles, the stochastic coalescent obviously exists without any assumption on K, and is known as the Marcus–Lushnikov process [12,11]. When there are initially infinitely many particles, stochastic coalescence with

* Corresponding author.
E-mail addresses: nicolas.fournier@univ-paris12.fr (N. Fournier), locherbach@univ-paris12.fr (E. Löcherbach).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2008.01.007

constant, additive, and multiplicative kernels have been extensively studied, see Kingman [10], Aldous–Pitman [2], Aldous [1]. In particular, Aldous showed in [1] that the stochastic coalescent with kernel $K(x, y) = xy$ exists as a Feller process in ℓ_2. These stochastic coalescents with special kernels have proved useful for application to biology, random graph theory, continuous random tree. . . .

We would like here to extend stochastic coalescence to more general kernels, as proposed by Aldous [3, Open Problem 13]. The main motivation to do this concerns the link between stochastic coalescence and the Smoluchowski coagulation, widely used in applied sciences (see the details in [3, Open Problems 14,15]): roughly, the stochastic coalescent, when started with infinitely many particles with very small masses, is expected to describe the long-time behaviour of any reasonable solution to the Smoluchowski equation. This is somewhat natural, since stochastic coalescence and the Smoluchowski equation describe the same phenomenon at different scales.

Another possible motivation concerns Monte Carlo methods for the Smoluchowski equation, for which the Marcus–Lushnikov process is widely used: our result will show some stability of the Marcus–Lushnikov processes with respect to its initial condition.

The first work dealing with the existence questions for general kernels seems to be that of Evans–Pitman [5]: they showed the existence of the stochastic coalescent as a Feller process with state space $\{m = (m_i)_{i\geq 1}, m_i \geq 0, \sum_{i\geq 1} im_i < \infty\}$ under the assumption that $K(x, y) \simeq x + y$ (that is, $K(0, 0) = 0$ and K is Lipschitz). In [6], kernels of the form $K(x, y) \simeq x^\lambda + y^\lambda$, for $\lambda \in [0, 1]$ (that is $|K(x, y) - K(u, y)| \leq C|x^\lambda - u^\lambda|$) are considered. The existence of the stochastic coalescent as a Feller process with state space ℓ_λ was proved.

In the present work, we pay particular attention to *homogeneous kernels*, which are kernels satisfying, for some *degree* $\lambda \in \mathbb{R}$, $K(xu, yu) = u^\lambda K(x, y)$. Such kernels are of particular importance in applications: Eight of the nine kernels presented in Aldous [3] Table 1 and taken from the physical literature are homogeneous.

We show that, for a class of homogeneous kernels (or having the same bounds and regularity as a homogeneous kernel), the stochastic coalescent exists as a Feller process with values in ℓ_λ, where $\lambda \in \mathbb{R}\setminus\{0\}$ is the degree of homogeneity of K. This is in agreement with Aldous [3], since the multiplicative kernel $K(x, y) = xy$ is homogeneous with degree 2.

This work is inspired by [8], where similar results were obtained for the deterministic Smoluchowski coagulation equation.

There has been a lot of work on the convergence of the stochastic coalescent to the Smoluchowski equation, when making the rate of coalescence tends to 0 as the number of particles tends to infinity, see Jeon [9], Norris [13], Fournier–Giet [7] and others. However, what we do here is very different, since we build stochastic coalescents with infinitely many particles, in which each pair of particles coalesces with a positive rate.

Observe also that no gelation (appearance of particles with infinite mass in finite time) occurs within our models, even with *gelling* kernels such as $K(x, y) = xy$ and with an initial condition with infinite total mass. This shows a qualitative difference between stochastic coalescence and the Smoluchowski equation.

2. Notation and main results

We first of all introduce the state spaces of our processes. We denote by $\mathcal{S}^\downarrow$ the set of non-increasing sequences $m = (m_n)_{n\geq 1}$ with values in $[0, \infty)$, and by $\mathcal{S}^\uparrow$ the set of non-decreasing sequences $m = (m_n)_{n \ge 1}$ with values in $(0, \infty]$. A state m in $\mathcal{S}^\uparrow$ or $\mathcal{S}^\downarrow$ represents the sequence of the ordered masses of the particles in a particle system. Next, for $\lambda \in \mathbb{R} \setminus \{0\}$, we consider

$$\ell_\lambda = \left\{ m = (m_k)_{k \ge 1} \in \mathcal{S}^\downarrow,\ \|m\|_\lambda := \sum_{k=1}^\infty m_k^\lambda < \infty \right\} \quad \text{if } \lambda > 0,$$

$$\ell_\lambda = \left\{ m = (m_k)_{k \ge 1} \in \mathcal{S}^\uparrow,\ \|m\|_\lambda := \sum_{k=1}^\infty m_k^\lambda < \infty \right\} \quad \text{if } \lambda < 0.$$

Note that for $\lambda > 0$ (resp. $\lambda < 0$), $m \in \ell_\lambda$ if m contains rather small (resp. large) particles.

Observe that ℓ_λ does not correspond to the usual ℓ_p spaces (because of the ordering requirements) and that $\|.\|_\lambda$ is not a norm. We however choose these notations for convenience.

We do not consider the case $\lambda = 0$, since this corresponds to finite particle systems.

We also consider the sets of finite particle systems, completed for convenience with infinitely many 0's or ∞'s according to the chosen ordering.

$$\ell_{0+} = \left\{ m = (m_k)_{k \ge 1} \in \mathcal{S}^\downarrow, \inf\{k \ge 1,\ m_k = 0\} < \infty \right\},$$

$$\ell_{0-} = \left\{ m = (m_k)_{k \ge 1} \in \mathcal{S}^\uparrow, \inf\{k \ge 1,\ m_k = \infty\} < \infty \right\}.$$

Observe that for all $0 < \lambda_1 < \lambda_2$, $\ell_{0+} \subset \ell_{\lambda_1} \subset \ell_{\lambda_2}$, while for all $\lambda_2 < \lambda_1 < 0$, $\ell_{0-} \subset \ell_{\lambda_1} \subset \ell_{\lambda_2}$.

Note also that if $\lambda \in (0, 1]$, for any $m \in \ell_\lambda$, the total mass $\sum_{k \ge 1} m_k$ of the system is finite, but it is not necessarily the case for $\lambda > 1$, and never the case when $\lambda < 0$.

For $i < j$, the coalescence between the ith and jth particles is described by the map $c_{ij} : \ell_\lambda \mapsto \ell_\lambda$, with

$$c_{ij}(m) = \text{reorder}(m_1, \ldots, m_{i-1}, m_i + m_j, m_{i+1}, \ldots, m_{j-1}, m_{j+1}, \ldots), \tag{2.1}$$

the reordering being in the decreasing (resp. increasing) order if $\lambda > 0$ or $\lambda = 0+$ (resp. $\lambda < 0$ or $\lambda = 0-$).

A coagulation kernel is a function K on $[0, \infty)^2$ such that $0 \le K(x, y) = K(y, x)$. In the whole paper, we will use the conventions that, when dealing with sequences in ℓ_λ,

$$\begin{aligned} &\text{if } \lambda < 0, \text{ or } \lambda = 0-, \quad K(x, \infty) = 0 \quad \text{for all } x \in (0, \infty], \\ &\text{if } \lambda > 0, \text{ or } \lambda = 0+, \quad K(x, 0) = 0 \quad \text{for all } x \in [0, \infty). \end{aligned} \tag{2.2}$$

Remark 2.1. Consider a coagulation kernel K. For any $m \in \ell_{0+}$ (resp. ℓ_{0-}), there obviously exists a unique (in law) strong Markov process $(M(m, t))_{t \ge 0}$ with values in ℓ_{0+} (resp. ℓ_{0-}), starting from $M(m, 0) = m$, with infinitesimal generator $\mathcal{L}$ defined, for all $\Phi : \ell_{0+} \mapsto \mathbb{R}$, (resp. $\Phi : \ell_{0-} \mapsto \mathbb{R}$), all $\mu \in \ell_{0+}$ (resp. $\mu \in \ell_{0-}$), by

$$\mathcal{L}\Phi(\mu) = \sum_{1 \le i < j < \infty} K(\mu_i, \mu_j) \left[\Phi(c_{ij}(\mu)) - \Phi(\mu) \right]. \tag{2.3}$$

The process $(M(m, t))_{t \ge 0}$ is known as the Marcus–Lushnikov process.

Notice that (2.3) is well defined for *all* functions Φ since the sum is actually finite thanks to (2.2). We refer to Aldous [3] for more details on this process.

We wish to extend this process to the case where the initial condition consists of infinitely many particles. To this aim, we will assume the following conditions, for some $\lambda \in \mathbb{R} \setminus \{0\}$. We set, for $x, y \in (0, \infty)$, $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$.

Assumption $A(\lambda)$.

Case 1: $\lambda < 0$. For all $\varepsilon > 0$, there exists a constant C_ε such that for all $x, y, \tilde{x}, \tilde{y} \in (\varepsilon, \infty)$,

$$K(x, y) \le C_\varepsilon (x + y)^\lambda, \tag{2.4}$$

$$\left[(x \wedge y)^\lambda + (\tilde{x} \wedge \tilde{y})^\lambda\right] |K(x, y) - K(\tilde{x}, \tilde{y})| \le C_\varepsilon \left[(x^\lambda + \tilde{x}^\lambda)|y^\lambda - \tilde{y}^\lambda| + (y^\lambda + \tilde{y}^\lambda)|x^\lambda - \tilde{x}^\lambda|\right]. \tag{2.5}$$

Case 2: $\lambda \in (0, 1]$. For all $a > 0$, there exists a constant C_a such that for all $x, y, \tilde{x}, \tilde{y} \in [0, a]$,

$$K(x, y) \le C_a (x + y)^\lambda, \tag{2.6}$$

$$\left[(x \wedge y)^\lambda + (\tilde{x} \wedge \tilde{y})^\lambda\right] |K(x, y) - K(\tilde{x}, \tilde{y})| \le C_a \left[(x^\lambda + \tilde{x}^\lambda)|y^\lambda - \tilde{y}^\lambda| + (y^\lambda + \tilde{y}^\lambda)|x^\lambda - \tilde{x}^\lambda|\right]. \tag{2.7}$$

Case 3: $\lambda \in (1, 2]$. There exists a constant C such that for all $x, y, \tilde{x}, \tilde{y} \in [0, \infty)$,

$$K(x, y) \le C (xy)^{\lambda/2}, \tag{2.8}$$

$$\left[(x \wedge y)(x \vee y)^{\lambda-1} + (\tilde{x} \wedge \tilde{y})(\tilde{x} \vee \tilde{y})^{\lambda-1}\right] |K(x, y) - K(\tilde{x}, \tilde{y})| \le C \left[(x^\lambda + \tilde{x}^\lambda)|y^\lambda - \tilde{y}^\lambda| + (y^\lambda + \tilde{y}^\lambda)|x^\lambda - \tilde{x}^\lambda|\right]. \tag{2.9}$$

Case 4: $\lambda \ge 2$. There exists a constant C such that for all $x, y, \tilde{x}, \tilde{y} \in [0, \infty)$,

$$K(x, y) \le C xy (x \wedge y)^{\lambda-2}, \tag{2.10}$$

$$\left[(x \wedge y)(x \vee y)^{\lambda-1} + (\tilde{x} \wedge \tilde{y})(\tilde{x} \vee \tilde{y})^{\lambda-1}\right] |K(x, y) - K(\tilde{x}, \tilde{y})| \le C \left[(x^\lambda + \tilde{x}^\lambda)|y^\lambda - \tilde{y}^\lambda| + (y^\lambda + \tilde{y}^\lambda)|x^\lambda - \tilde{x}^\lambda|\right]. \tag{2.11}$$

These assumptions seem to be the best we can treat with our methods. Only (2.6) could be removed as in [6], but we decide to keep it in order to unify the proofs.

For example, the kernels listed below, taken from the mathematical and physical literature, satisfy $A(\lambda)$. Many of them (with explicit values for the parameters) can be found in Aldous [3, Table 1] and Drake [4, Section 4.3]

$$K(x, y) = (x^\lambda \wedge y^\lambda),\ \lambda \in \mathbb{R} \setminus \{0\},$$
$$K(x, y) = (x^\alpha + y^\alpha)^\beta,\ \alpha > 0,\ \beta \in \mathbb{R},\ \lambda = \alpha\beta \in (-\infty, 1] \setminus \{0\},$$
$$K(x, y) = (xy)^{\lambda/2},\ \lambda \in (0, 2],$$
$$K(x, y) = (xy)^\alpha (x + y)^{-\beta},\ \alpha > 0,\ \beta \ge 0,\ \lambda = 2\alpha - \beta \in (-\infty, 2] \setminus \{0\},$$
$$K(x, y) = (xy)^\alpha (x + y)^{-\beta},\ \beta > 0,\ 0 < \alpha \le \beta + 1,\ \lambda = 2\alpha - \beta \in (2, \infty),$$
$$K(x, y) = (x^\alpha + y^\alpha)^\beta |x^\gamma - y^\gamma|,\ \alpha > 0,\ \beta > 0,\ \gamma \in (0, 1],\ \lambda = \alpha\beta + \gamma \in (0, 1],$$
$$K(x, y) = |x - y|^\alpha (x + y)^{-\beta},\ \alpha \ge 1,\ \beta > 0,\ \lambda = \alpha - \beta \in (-\infty, 1] \setminus \{0\},$$
$$K(x, y) = (x^{1/3} + y^{1/3})(xy)^{1/2}(x + y)^{-3/2},\ \lambda = -1/6.$$

To state our main result, we finally need to introduce some notation: for $\lambda \in \mathbb{R} \setminus \{0\}$, and for $m, \tilde{m} \in \ell_\lambda$, we consider the distance

$$d_\lambda(m, \tilde{m}) = \sum_{k \geq 1} \left| m_k^\lambda - \tilde{m}_k^\lambda \right|, \tag{2.12}$$

with the natural convention $\infty^\lambda = 0$ if $\lambda < 0$. When $\lambda \in (-\infty, 0) \cup (0, 1]$, this specific distance enjoys the property that for any pair of states $m, \tilde{m} \in \ell_\lambda$, any $i < j$, $d_\lambda(c_{ij}(m), c_{ij}(\tilde{m})) \leq d_\lambda(m, \tilde{m})$ (see (A.5) and (A.8)): it decreases under simultaneous coalescence.

Notice that for m^n, m in ℓ_λ,

$$\lim_n d_\lambda(m^n, m) = 0 \Longleftrightarrow \begin{cases} \lim_n \sum_{i \geq 1} |m_i^n - m_i|^\lambda = 0 & \text{if } \lambda > 0, \\ \lim_n \sum_{i \geq 1} \left| \frac{1}{m_i^n} - \frac{1}{m_i} \right|^{|\lambda|} = 0 & \text{if } \lambda < 0. \end{cases} \tag{2.13}$$

Our main result is as follows.

Theorem 2.2. *Let $\lambda \in \mathbb{R} \setminus \{0\}$, consider a coagulation kernel satisfying $A(\lambda)$, and endow ℓ_λ with the distance d_λ.*

(i) *For any $m \in \ell_\lambda$, there exists a (necessarily unique in law) strong Markov process $(M(m,t))_{t \geq 0} \in \mathbb{D}([0,\infty), \ell_\lambda)$ enjoying the following property. For any sequence $m^n \in \ell_{0+}$ (if $\lambda > 0$) or $m^n \in \ell_{0-}$ (if $\lambda < 0$) such that $\lim_{n\to\infty} d_\lambda(m^n, m) = 0$, the sequence of Marcus–Lushnikov processes $(M(m^n, t))_{t \geq 0}$ converges in law, in $\mathbb{D}([0,\infty), \ell_\lambda)$, to $(M(m,t))_{t \geq 0}$.*

(ii) *The obtained process is Feller in the sense that for all $t \geq 0$, the map $m \mapsto Law(M(m,t))$ is continuous from ℓ_λ into $\mathcal{P}(\ell_\lambda)$.*

(iii) *For all bounded $\Phi : \ell_\lambda \mapsto \mathbb{R}$ satisfying $|\Phi(m) - \Phi(\tilde{m})| \leq a d_\lambda(m, \tilde{m})$ for some constant $a \geq 0$, the process*

$$\Phi(M(m,t)) - \Phi(m) - \int_0^t \mathrm{d}s \sum_{1 \leq i < j < \infty} K(M_i(m,s), M_j(m,s)) \\ \times \left[\Phi(c_{ij}(M(m,s))) - \Phi(M(m,s)) \right] \tag{2.14}$$

is a martingale if $\lambda < 0$ or $\lambda \in (0,1]$, and a local martingale if $\lambda > 1$.

For $\lambda < 0$, the fact that $M(m,t) \in \ell_\lambda$ does not imply that $M_k(m,t) < \infty$ for all $k \geq 1$. However, the appearance of infinite particles does not occur with our assumptions.

Proposition 2.3. *Let $\lambda < 0$, consider $m \in \ell_\lambda$, a coagulation kernel K satisfying $A(\lambda)$ and $K(x,y) > 0$ for all $x, y \in (0,\infty)$, and the Markov process $(M(m,t))_{t \geq 0}$ built in Theorem 2.2. If $m \in \ell_\lambda \setminus \ell_{0-}$, then a.s., for all $t \geq 0$, $M(m,t) \in \ell_\lambda \setminus \ell_{0-}$. In other words, if $m_k < \infty$ for all $k \geq 1$, then a.s., $M_k(m,t) < \infty$ for all $t \geq 0$, all $k \geq 1$.*

The condition $K > 0$ is assumed only to simplify the proof.

The rest of the paper is devoted to the proofs of Theorem 2.2 and Proposition 2.3. A Poisson-driven stochastic differential equation is introduced in Section 3, which allows us to couple in a convenient way the two stochastic coalescents starting from two different initial conditions. Using this coupling, we show in Section 4 that for $m, \tilde{m} \in \ell_\lambda$, $d_\lambda(M(m,t), M(\tilde{m},t))$ cannot increase too much with time, at least while $\|M(m,t)\|_\lambda$ and $\|M(\tilde{m},t)\|_\lambda$ remain finite. We introduce another Poisson interpretation of Marcus–Lushnikov processes in Section 5, which allows us to establish in Sections 6 and 7 that $\|M(m,t)\|_\lambda$ remains finite for all times. We prove, in Section 8, the existence and uniqueness of a solution to the S.D.E. introduced in Section 3. We conclude all the proofs in Section 9. Finally, an appendix contains inequalities concerning the action of coalescence on the distance d_λ and on the moment $\|.\|_\lambda$.

3. A Poisson-driven S.D.E.

We now introduce a representation of stochastic coalescents in terms of Poisson measures, in order to couple stochastic coalescents with different initial data.

Definition 3.1. Let $\lambda \in \mathbb{R} \setminus \{0\}$ be fixed and assume $A(\lambda)$. Endow ℓ_λ with the distance d_λ. Let $N(\mathrm{d}t, \mathrm{d}(i,j), \mathrm{d}z)$ be a Poisson measure on $[0,\infty) \times \{(i,j) \in \mathbb{N}^2 : i < j\} \times [0,\infty)$ with intensity measure $\mathrm{d}t(\sum_{k<l} \delta_{(k,l)})\mathrm{d}z$, and let $(\mathcal{F}_t)_{t\ge 0}$ stand for the associated canonical filtration.

For $m \in \ell_\lambda$, a $(\mathcal{F}_t)_{t\ge0}$-adapted process $(M(m,t))_{t\ge0}$ is said to be a solution to $(SDE)(\lambda, m, K, N)$ if it a.s. belongs to $\mathbb{D}([0,\infty), \ell_\lambda)$ and if for all $k \ge 1$, all $t \ge 0$, a.s.

$$[M_k(m,t)]^\lambda = [m_k]^\lambda + \int_0^t \int_{i<j} \int_0^\infty \{[c_{ij}(M(m,s-))]_k^\lambda - [M_k(m,s-)]^\lambda\} \times 1_{\{z \le K[M_i(m,s-), M_j(m,s-)]\}} N(\mathrm{d}s, \mathrm{d}(i,j), \mathrm{d}z). \tag{3.1}$$

Observe that it would be more natural (and it is of course equivalent in some sense) to write this equation without the power λ. However, it will ensure that the integral on the right-hand side of (3.1) is well defined.

Remark 3.2. Let $m \in \ell_{0+}$ (resp. $m \in \ell_{0-}$). Consider a Poisson measure N as in Definition 3.1. Then there exists a unique process $(M(m,t))_{t\ge0}$ which solves $(SDE)(\lambda, m, K, N)$ for all $\lambda > 0$ (resp. all $\lambda < 0$). This process is a Marcus–Lushnikov process with initial condition m and coagulation kernel K.

This remark is straightforward, since in such a case, the total rate of jumps of the system is uniformly bounded. We now check that the integral in (3.1) always makes sense.

Lemma 3.3. *Let $\lambda \in \mathbb{R} \setminus \{0\}$, assume $A(\lambda)$, consider a Poisson measure as in Definition 3.1 and any $(\mathcal{F}_t)_{t\ge0}$-adapted process $(M(t))_{t\ge0}$ belonging a.s. to $\mathbb{D}([0,\infty), \ell_\lambda)$. Then a.s.,*

$$\int_0^t \int_{i<j} \int_0^\infty [(c_{ij}(M(s-)))_k^\lambda - M_k(s-)^\lambda] 1_{\{z \le K(M_i(s-), M_j(s-))\}} N(\mathrm{d}s, \mathrm{d}(i,j), \mathrm{d}z)$$

is well defined and finite for all $k \ge 1$, $t \ge 0$.

Proof. The processes in the integral being càdlàg and adapted, it suffices to check that a.s., the compensators are a.s. finite. In other words, we just have to show that a.s., for all $k \ge 1$, all $t \ge 0$,

$$C_k(t) := \int_0^t \mathrm{d}s \sum_{i<j} K(M_i(s), M_j(s)) \left|[c_{ij}(M(s))]_k^\lambda - M_k(s)^\lambda\right| < \infty. \tag{3.2}$$

To do so, we consider

$$A_t := \sum_{k\ge1} C_k(t) = \int_0^t \mathrm{d}s \sum_{i<j} K(M_i(s), M_j(s)) d_\lambda(c_{ij}(M(s)), M(s)). \tag{3.3}$$

We now consider the different cases separately.

Case 1: $\lambda < 0$. Since M belongs a.s. to $\mathbb{D}([0,\infty),\ell_\lambda)$, $\varepsilon_t := \inf_{[0,t]} M_1(s) > 0$. Thus for all $i \geq 1$, all $s \in [0,t]$, $M_i(s) \geq \varepsilon_t$. Applying (2.4) and (A.4), we obtain

$$\begin{aligned} A_t &\leq \int_0^t \mathrm{d}s \sum_{i<j} 2C_\varepsilon (M_i(s)+M_j(s))^\lambda M_i(s)^\lambda \\ &\leq 2C_{\varepsilon_t} \int_0^t \mathrm{d}s \sum_{i<j} M_j(s)^\lambda M_i(s)^\lambda \leq 2C_{\varepsilon_t} t \sup_{[0,t]} \|M(s)\|_\lambda^2 < \infty, \end{aligned} \tag{3.4}$$

since M belongs a.s. to $\mathbb{D}([0,\infty),\ell_\lambda)$.

Case 2: $\lambda \in (0,1]$. Notice first that for all $s \in [0,t]$, all $i \geq 1$, $M_i(s) \leq \|M(s)\|_\lambda^{1/\lambda} \leq \sup_{[0,t]} \|M(s)\|_\lambda^{1/\lambda} =: a_t < \infty$ a.s., since M belongs a.s. to $\mathbb{D}([0,\infty),\ell_\lambda)$. Thus using (2.6) and (A.7), we deduce, since $M_j(s) \leq M_i(s)$, that

$$\begin{aligned} A_t &\leq \int_0^t \mathrm{d}s \sum_{i<j} 2C_{a_t} (M_i(s)+M_j(s))^\lambda M_j(s)^\lambda \\ &\leq 2^{\lambda+1} C_{a_t} t \sup_{[0,t]} \sum_{i<j} M_i(s)^\lambda M_j(s)^\lambda \leq 2^{\lambda+1} C_{a_t} t \sup_{[0,t]} \|M(s)\|_\lambda^2 < \infty. \end{aligned} \tag{3.5}$$

Case 3: $\lambda \in (1,2]$. Using (2.8) and (A.10), we get, since $M_j(s) \leq M_i(s)$,

$$\begin{aligned} A_t &\leq \int_0^t \mathrm{d}s \sum_{i<j} (1+2^{\lambda-1})\lambda C (M_i(s)M_j(s))^{\lambda/2} M_j(s) M_i(s)^{\lambda-1} \\ &\leq (1+2^{\lambda-1})\lambda C \int_0^t \mathrm{d}s \sum_{i<j} M_i(s)^\lambda M_j(s)^\lambda \\ &\leq (1+2^{\lambda-1})\lambda C t \sup_{[0,t]} \|M(s)\|_\lambda^2 < \infty. \end{aligned} \tag{3.6}$$

Case 4: $\lambda > 2$. Using (2.10) and (A.10), we obtain, since $M_j(s) \leq M_i(s)$,

$$\begin{aligned} A_t &\leq \int_0^t \mathrm{d}s \sum_{i<j} (1+2^{\lambda-1})\lambda C M_i(s) M_j(s)^{\lambda-1} M_j(s) M_i(s)^{\lambda-1} \\ &\leq (1+2^{\lambda-1})\lambda C t \sup_{[0,t]} \|M(s)\|_\lambda^2 < \infty, \end{aligned} \tag{3.7}$$

since M belongs a.s. to $\mathbb{D}([0,\infty),\ell_\lambda)$. □

4. A Gronwall-type inequality

We now check a fundamental inequality, which shows in some sense that the distance d_λ between two coalescents cannot increase too much, at least while their moments of order λ stay finite.

Proposition 4.1. *Let $\lambda \in \mathbb{R} \setminus \{0\}$, and assume $A(\lambda)$. Consider a Poisson measure N as in Definition 3.1 and $m, \tilde{m} \in \ell_\lambda$. Assume that there exist solutions $M(m,t)$ and $M(\tilde{m},t)$ to $SDE(\lambda,m,K,N)$ and $SDE(\lambda,\tilde{m},K,N)$.*

Case 1*: $\lambda < 0$. The map $t \mapsto \|M(m,t)\|_\lambda$ is a.s. non-increasing, while $t \mapsto M_1(m,t)$ is a.s. non-decreasing. For all $t \geq 0$,*

$$E\left[\sup_{[0,t]} d_\lambda(M(m,s), M(\tilde{m},s))\right] \leq d_\lambda(m,\tilde{m})\mathrm{e}^{8C_{m_1\wedge\tilde{m}_1}(\|m\|_\lambda+\|\tilde{m}\|_\lambda)t}, \tag{4.1}$$

where C_ε was defined in $A(\lambda)$.

Case 2*: $\lambda \in (0,1]$. The maps $t \mapsto \|M(m,t)\|_\lambda$ and $t \mapsto \|M(m,t)\|_1$ are a.s. non-increasing. For all $t \geq 0$,*

$$E\left[\sup_{[0,t]} d_\lambda(M(m,s), M(\tilde{m},s))\right] \leq d_\lambda(m,\tilde{m})\mathrm{e}^{8C_{\|m\|_1\vee\|\tilde{m}\|_1}(\|m\|_\lambda+\|\tilde{m}\|_\lambda)t}, \tag{4.2}$$

where C_a was defined in $A(\lambda)$.

Case 3*: $\lambda > 1$. The map $t \mapsto \|M(m,t)\|_\lambda$ is a.s. non-decreasing. Define, for all $x > 0$, the stopping time $\tau(m,x) = \inf\{t \geq 0,\ \|M(m,t)\|_\lambda \geq x\}$. Then for any $t \geq 0$, any $x > 0$,*

$$E\left[\sup_{[0,t\wedge\tau(m,x)\wedge\tau(\tilde{m},x))} d_\lambda(M(m,s), M(\tilde{m},s))\right] \leq d_\lambda(m,\tilde{m})\mathrm{e}^{Cxt} \tag{4.3}$$

where C is a constant depending only on K and λ.

Proof. We write $M(t) := M(m,t)$ and $\tilde{M}(t) = M(\tilde{m},t)$ for simplicity. In all cases, since M and $\tilde{M}$ solve (3.1) with the same Poisson measure, we have

$$d_\lambda(M(t),\tilde{M}(t)) = d_\lambda(m,\tilde{m}) + A_t + B_t + C_t, \tag{4.4}$$

where

$$\begin{aligned}
A_t &:= \int_0^t \int_{i<j}\int_0^\infty \{d_\lambda(c_{ij}(M(s-)), c_{ij}(\tilde{M}(s-))) - d_\lambda(M(s-),\tilde{M}(s-))\} \\
&\quad \times 1_{\{z\leq K(M_i(s-),M_j(s-))\wedge K(\tilde{M}_i(s-),\tilde{M}_j(s-))\}} N(\mathrm{d}s,\mathrm{d}(i,j),\mathrm{d}z), \\
B_t &:= \int_0^t \int_{i<j}\int_0^\infty \{d_\lambda(c_{ij}(M(s-)), \tilde{M}(s-)) - d_\lambda(M(s-),\tilde{M}(s-))\} \\
&\quad \times 1_{\{K(\tilde{M}_i(s-),\tilde{M}_j(s-))\leq z\leq K(M_i(s-),M_j(s-))\}} N(\mathrm{d}s,\mathrm{d}(i,j),\mathrm{d}z), \\
C_t &:= \int_0^t \int_{i<j}\int_0^\infty \{d_\lambda(c_{ij}(\tilde{M}(s-)), M(s-)) - d_\lambda(M(s-),\tilde{M}(s-))\} \\
&\quad \times 1_{\{K(M_i(s-),M_j(s-))\leq z\leq K(\tilde{M}_i(s-),\tilde{M}_j(s-))\}} N(\mathrm{d}s,\mathrm{d}(i,j),\mathrm{d}z).
\end{aligned} \tag{4.5}$$

Also note that in any case,

$$|d_\lambda(c_{ij}(M(s-)),\tilde{M}(s-)) - d_\lambda(M(s-),\tilde{M}(s-))| \leq d_\lambda(c_{ij}(M(s-)), M(s-)). \tag{4.6}$$

Case 1: $\lambda < 0$. The fact that $t \mapsto \|M(t)\|_\lambda$ is non-increasing follows from (A.3), while $t \mapsto M_1(t)$ is non-decreasing since for all $m \in \ell_\lambda$, all $i < j$, $[c_{ij}(m)]_1 \geq m_1$. We deduce from this last property that for all $t \geq 0$, all $i \geq 1$, $M_i(t) \geq m_1$ while $\tilde{M}_i(t) \geq \tilde{m}_1$. First, we obtain immediately from (A.5) that a.s., $A_t \leq 0$ for all $t \geq 0$. Next, setting $\varepsilon := m_1 \wedge \tilde{m}_1$, we deduce from (4.6), (A.4) and (2.5) that

$$\begin{aligned}
E\left[\sup_{[0,t]}|B_s|\right] &\le \int_0^t E\left[\sum_{i<j} 2M_i^\lambda(s)\left|K(M_i(s), M_j(s)) - K(\tilde{M}_i(s), \tilde{M}_j(s))\right|\right]\mathrm{d}s \\
&\le 2C_\varepsilon \int_0^t E\left[\sum_{i<j}(M_i(s)^\lambda + \tilde{M}_i(s)^\lambda)|M_j(s)^\lambda - \tilde{M}_j(s)^\lambda|\right. \\
&\qquad \left. + (M_j(s)^\lambda + \tilde{M}_j(s)^\lambda)|M_i(s)^\lambda - \tilde{M}_i(s)^\lambda|\right]\mathrm{d}s \\
&\le 4C_\varepsilon \int_0^t E\left[(\|M_s\|_\lambda + \|\tilde{M}_s\|_\lambda) d_\lambda(M(s), \tilde{M}(s))\right]\mathrm{d}s \\
&\le 4C_\varepsilon(\|m\|_\lambda + \|\tilde{m}\|_\lambda)\int_0^t E\left[d_\lambda(M(s), \tilde{M}(s))\right]\mathrm{d}s. \qquad (4.7)
\end{aligned}$$

Using the same evaluation for C_t, we get

$$\begin{aligned}
E\left[\sup_{[0,t]} d_\lambda(M(s), \tilde{M}(s))\right] &\le d_\lambda(m, \tilde{m}) \\
&\quad + 8C_\varepsilon(\|m\|_\lambda + \|\tilde{m}\|_\lambda)\int_0^t E\left[d_\lambda(M(s), \tilde{M}(s))\right]\mathrm{d}s. \qquad (4.8)
\end{aligned}$$

Then the Gronwall lemma allows us to conclude.

Case 2: $\lambda \in (0, 1]$. We obtain, using (A.6) that a.s., $t \mapsto \|M(t)\|_\lambda$ and $t \mapsto \|M(t)\|_1$ are non-increasing. We deduce that setting $a := \|m\|_1 \vee \|\tilde{m}\|_1$, we have for all $t \ge 0$ and all $i \ge 1$, $M_i(t) \le a$ and $\tilde{M}_i(t) \le a$. We have as previously (due to (A.8)) $A_t \le 0$ for all $t \ge 0$. Copying line by line the computation handled in (4.7), using of course (2.7) and (A.7), we obtain (4.8) with C_a instead of C_ε, and we conclude as previously.

Case 3: $\lambda \in (1, 2]$. First of all, $t \mapsto \|M(t)\|_\lambda$ is a.s. non-decreasing, since $(m_i+m_j)^\lambda \ge m_i^\lambda + m_j^\lambda$. We set $\tau_x = \tau(m, x) \wedge \tau(\tilde{m}, x)$ for simplicity. We denote by C any constant depending only on λ and K. Due to (A.11),

$$\begin{aligned}
E\left[\sup_{[0,t\wedge\tau_x)} A_s\right] &\le E\left[\int_0^{t\wedge\tau_x}\sum_{i<j} C\left[(M_i(s-)^\lambda + \tilde{M}_i(s-)^\lambda)|M_j(s-)^\lambda - \tilde{M}_j(s-)^\lambda|\right.\right. \\
&\qquad \left.\left. + \ (M_j(s-)^\lambda + \tilde{M}_j(s-)^\lambda)|M_i(s-)^\lambda - \tilde{M}_i(s-)^\lambda|\right]\mathrm{d}s\right] \\
&\le CE\left[\int_0^{t\wedge\tau_x}\left(\|M(s-)\|_\lambda + \|\tilde{M}(s-)\|_\lambda\right) d_\lambda(M(s-), \tilde{M}(s-))\mathrm{d}s\right] \\
&\le Cx\int_0^t E\left[\sup_{[0,s\wedge\tau_x)} d_\lambda(M(u), \tilde{M}(u))\right]\mathrm{d}s. \qquad (4.9)
\end{aligned}$$

By the same way, using (4.6), (A.10) and (2.9),

$$E\left[\sup_{[0,t\wedge\tau_x)}|B_s|\right] \le CE\left[\int_0^{t\wedge\tau_x}\sum_{i<j}\{M_j(s-)M_i(s-)^{\lambda-1}\right.$$

$$\left. \times\ |K(M_i(s-), M_j(s-)) - K(\tilde{M}_i(s-), \tilde{M}_j(s-))|\}\mathrm{d}s \right]$$

$$\leq CE\left[\int_0^{t\wedge\tau_x} \sum_{i<j} (M_i(s-)^\lambda + \tilde{M}_i(s-)^\lambda)|M_j(s-)^\lambda - \tilde{M}_j(s-)^\lambda| \right.$$

$$\left. +\ (M_j(s-)^\lambda + \tilde{M}_j(s-)^\lambda)|M_i(s-)^\lambda - \tilde{M}_i(s-)^\lambda|\mathrm{d}s \right]$$

$$\leq Cx \int_0^t E\left[\sup_{[0,s\wedge\tau_x)} d_\lambda(M(u), \tilde{M}(u))\right] \mathrm{d}s. \tag{4.10}$$

We use the same computation for C_t and get finally that

$$E\left[\sup_{[0,t\wedge\tau_x)} d_\lambda(M(s), \tilde{M}(s))\right] \leq d_\lambda(m, \tilde{m}) + Cx \int_0^t E\left[\sup_{[0,s\wedge\tau_x)} d_\lambda(M(u), \tilde{M}(u))\right] \mathrm{d}s. \tag{4.11}$$

We conclude with the Gronwall Lemma.

Finally, the case where $\lambda > 2$ is handled as Case 3, since (A.10), (A.11) and (2.11) allow us to get exactly the same estimates for A_t, B_t and C_t. $\square$

5. Another Poisson representation

While we can clearly conclude from Proposition 4.1 the existence and uniqueness of a solution to $(SDE)(\lambda, m, K, N)$ for $\lambda \in (-\infty, 1] \setminus \{0\}$ and $m \in \ell_\lambda$, we clearly have to show that when $\lambda > 1$, the moment of order λ of the solution does not explode. The first idea is to estimate, for example, $E[\|M(m,t)\|_\lambda]$. Easy considerations show that *a priori*, for example for $\lambda = 2$ and $K(x, y) = xy$,

$$\frac{\mathrm{d}}{\mathrm{d}t} E[\|M(m,t)\|_2] = CE\left[\sum_{i<j} M_i(m,t)^2 M_j(m,t)^2\right] \simeq CE\left[\|M(m,t)\|_2^2\right]. \tag{5.1}$$

This implies, if the "$\simeq$" is correct, that this expectation does not stay finite for all times. Similar considerations suggest that $E[\log \|M(m,t)\|_2]$ may not remain finite for all times. We thus have to understand more precisely the behaviour of our coalescents. We will thus consider a more complete description, which keeps track of the sets of particles that have coagulated.

We start with a given initial condition $m \in \mathcal{S}^\downarrow$ (or $m \in \mathcal{S}^\uparrow$). We label 1 the particle with mass m_1, 2 the particle with mass m_2, and so on. We wish to build a process $(Z(t))_{t\geq 0}$, taking its values in the state space

$$\mathcal{P} := \{(p_n)_{n\geq 1}, \forall\, n \geq 1,\ p_n \subset \mathbb{N},\ n \in p_n \text{ and } \forall k \in p_n,\ p_n = p_k\}, \tag{5.2}$$

and such that $Z_k(t)$ represents the set of the labels of the particles which are in the same cluster, at time t, as that labelled k.

For $1 \leq i < j$, the coalescence of the clusters containing i and j is described by the map $\mathbf{c}_{i,j} : \mathcal{P} \mapsto \mathcal{P}$ (in bold) defined by

$$\forall k \in p_i \cup p_j, \quad [\mathbf{c}_{i,j}(p)]_k := p_i \cup p_j \quad \text{and} \quad \forall k \notin p_i \cup p_j, \quad [\mathbf{c}_{i,j}(p)]_k := p_k. \tag{5.3}$$

Notice that for any $p \in \mathcal{P}$, any $1 \leq i < j$, if $k \in p_i$ and $l \in p_j$, $\mathbf{c}_{k,l}(p) = \mathbf{c}_{i,j}(p)$.

Note also that if $k \in p_l$ (i.e. $p_k = p_l$), $\mathbf{c}_{k,l}(p) = p$.

For a given (initial condition) $m \in \mathcal{S}^{\downarrow}$ (or $m \in \mathcal{S}^{\uparrow}$), any $p \in \mathcal{P}$, any $k \geq 1$, we define $\mu_k(m, p) := \sum_{i \in p_k} m_i$, which represents the mass of the cluster containing k. Clearly, if $i \in p_k$, then $\mu_k(m, p) = \mu_i(m, p)$.

For a given $p \in \mathcal{P}$, we associate a partition $\xi(p)$ of $\mathbb{N}$, consisting of the blocks $\xi_1(p) = p_{i_1}$, $\xi_2(p) = p_{i_2}$, $\xi_3(p) = p_{i_3}, \ldots$, where $i_1 = 1$ and for $k \geq 2$,

$$i_k := \min(l \geq 1 : l \notin \cup_{j=1}^{k-1} p_{i_j}) \tag{5.4}$$

and we set

$$\mathcal{M}(m, p) := \text{reorder}((\mu_{i_k}(m, p))_{k \geq 1}). \tag{5.5}$$

Of course, the reordering is in the non-increasing (resp. non-decreasing) sense if $m \in \mathcal{S}^{\downarrow}$ (resp. $m \in \mathcal{S}^{\uparrow}$).

Notice that for any $\lambda > 1$, $m \in \ell_\lambda$, $p \in \mathcal{P}$

$$\|\mathcal{M}(m, p)\|_\lambda = \sum_{k \geq 1} [\mu_{i_k}(m, p)]^\lambda = \sum_{i \geq 1} m_i [\mu_i(m, p)]^{\lambda - 1}. \tag{5.6}$$

The following remark describes in a convenient way Marcus–Lushnikov processes.

Remark 5.1. Let $m \in \ell_{0+}$ (resp. ℓ_{0-}), let K be any coagulation kernel. Consider a Poisson measure $O(\mathrm{d}t, \mathrm{d}(k, l), \mathrm{d}z)$ on $[0, \infty) \times \{(k, l) \in \mathbb{N}^2,\ k < l\} \times [0, \infty)$ with intensity measure $\mathrm{d}t (\sum_{k<l} \delta_{(k,l)}) \mathrm{d}z$, denote by $(\mathcal{F}_t)_{t \geq 0}$ the associated canonical filtration.

Consider a (deterministic) non-negative function $f(k, l, p)$ on $\{k < l\} \times \mathcal{P}$, depending possibly on m, such that for all $k < l$, all $p \in \mathcal{P}$,

$$p_k \neq p_l \Rightarrow \sum_{i<j} 1_{\{i \in p_k,\ j \in p_l \text{ or } j \in p_k,\ i \in p_l\}} f(i, j, p) = 1. \tag{5.7}$$

There exists a unique $(\mathcal{F}_t)_{t \geq 0}$-adapted $\mathcal{P}$-valued process $(Z(t))_{t \geq 0}$ such that

$$\begin{aligned} Z(t) = Z(0) + \int_0^t \int_{k<l} \int_0^\infty & \{\mathbf{c}_{k,l}(Z(s-)) - Z(s-)\} \\ & \times 1_{\{z \leq K[\mu_k(m, Z(s-)), \mu_l(m, Z(s-))] f(k, l, Z(s-))\}} O(\mathrm{d}s, \mathrm{d}(k, l), \mathrm{d}z), \end{aligned} \tag{5.8}$$

where $Z_0 = (\{1\}, \{2\}, \{3\}, \ldots)$.

Furthermore, $(\mathcal{M}(m, Z(t)))_{t \geq 0}$ is a Marcus–Lushnikov process with initial condition m and coagulation kernel K.

Of course, it makes *in general* no sense to add sets. However, we write the integral in (5.8) in the sense that at each time of jump s with *marks* $k < l$, $Z(s) = Z(s-) + (\mathbf{c}_{k,l}(Z(s-)) - Z(s-)) = \mathbf{c}_{k,l}(Z(s-))$.

We will for example make use of the choice $f(k, l, p) = \frac{m_k m_l}{\mu_k(m,p) \mu_l(m,p)}$: for all $k < l$, if $p_k \neq p_l$, then $p_k \cap p_l = \emptyset$, so that $\sum_{i<j} 1_{\{i \in p_k, j \in p_l \text{ or } j \in p_k,\ i \in p_l\}} m_i m_j = (\sum_{i \in p_k} m_i)(\sum_{j \in p_l} m_j) = \mu_k(m, p) \mu_l(m, p)$.

The existence and uniqueness of Z is obvious again, since we deal here with finite particle systems: for example, if $m = (m_1, \ldots, m_n, 0, \ldots) \in \ell_{0+}$, then $K(\mu_k(m, Z(s-)), \mu_l(m, Z(s-))) = 0$ for all $s \geq 0$ as soon as $k > n$ or $l > n$. To understand that $(\mathcal{M}(m, Z(t)))_{t \geq 0}$ is a Marcus–Lushnikov process, note that $\mathcal{M}(m, Z(0)) = m$, and that any pair of clusters with masses x and y merge at total rate $K(x,y)$. For example, at time t, $Z_1(t)$ and $Z_2(t)$ merge, if they have not done so already (that is if $Z_1(t) \neq Z_2(t)$), at rate

$$\sum_{k<l} 1_{\{k\in Z_1(t), l\in Z_2(t) \text{ or } l\in Z_1(t), k\in Z_2(t)\}} K[\mu_k(m, Z(t)), \mu_l(m, Z(t))] f(k,l,Z(t))$$
$$= K(x,y) \sum_{k<l} 1_{\{k\in Z_1(t), l\in Z_2(t) \text{ or } l\in Z_1(t), k\in Z_2(t)\}} f(k,l,Z(t)) = K(x,y)$$

due to (5.7), with the notation $x := \mu_1(m, Z(t)) = \mu_k(m, Z(t))$ for all $k \in Z_1(t)$ and $y := \mu_2(m, Z(t)) = \mu_l(m, Z(t))$ for all $l \in Z_2(t)$.

6. Finiteness when $\lambda \in (1, 2]$

To show that when $\lambda \in (1, 2]$, the moment of order λ of the solution does not explode, we show that we may upperbound, in some sense, our coalescent by a multiplicative coalescent. Then we will use the following results of Aldous [1, Proposition 5].

Theorem 6.1. *Assume that* $K(x,y) = xy$.

(i) *For all* $\mathbf{x} \in \ell_2$, *there exists a (necessarily unique in law) strong Markov process* $(X(\mathbf{x},t))_{t\geq 0} \in \mathbb{D}([0,\infty), \ell_2)$ *enjoying the following property. For any sequence* $\mathbf{x}^n \in \ell_{0+}$ *satisfying the condition* $\lim_{n\to\infty} d_2(\mathbf{x}^n, \mathbf{x}) = 0$, *the sequence of Marcus–Lushnikov processes* $(X(\mathbf{x}^n,t))_{t\geq 0}$ *with kernel* K *and initial condition* $\mathbf{x}^n$ *converges in law to* $(X(\mathbf{x},t))_{t\geq 0}$ *in* $\mathbb{D}([0,\infty), \ell_2)$.

(ii) *The obtained process is Feller in the sense that for all* $t \geq 0$, *the application* $\mathbf{x} \mapsto Law(X(\mathbf{x},t))$ *is continuous from* ℓ_2 *into* $\mathcal{P}(\ell_2)$.

(iii) *The map* $t \mapsto \|X(\mathbf{x},t)\|_2$ *is a.s. non-decreasing.*

This section is devoted to the proof of the following result, which shows that *a priori*, $\|M(m,t)\|_\lambda$ does not explode, uniformly for m in a suitable set.

Theorem 6.2. *Let* $\lambda \in (1,2]$ *be fixed, and consider a coagulation kernel* $K(x,y)$ *satisfying the sole assumption* $K(x,y) \leq C(xy)^{\lambda/2}$. *Consider a subset* $\mathcal{A}$ *of* ℓ_{0+} *such that* $\sup_{m\in\mathcal{A}} \|m\|_\lambda < \infty$ *and* $\lim_{i\to\infty} \sup_{m\in\mathcal{A}} \sum_{k\geq i} m_k^\lambda = 0$. *Then for each* $t \geq 0$, $\lim_{x\to\infty} \alpha(t,x) = 0$, *where*

$$\alpha(t,x) := \sup_{m\in\mathcal{A}} P\left[\sup_{s\in[0,t]} \|M(m,s)\|_\lambda \geq x\right], \tag{6.1}$$

where $(M(m,t))_{t\geq 0}$ *stands for the Marcus–Lushnikov process starting from m with coagulation kernel* K.

We will essentially use this theorem in the following context: if a sequence $m^n \in \ell_{0+}$ satisfies $\lim_n d_\lambda(m^n, m) = 0$ for some $m \in \ell_\lambda$, then the set $\mathcal{A} := \{m^n, n \geq 1\}$ fulfills the required conditions, so that the stopping times $\tau(m^n, x)$ tend to infinity as $x \to \infty$, uniformly in n.

To prove this result, we first compare our Marcus–Lushnikov process with a suitable multiplicative coalescent.

Lemma 6.3. *Let* $\lambda \in (1,2]$ *be fixed, and consider a coagulation kernel* $K(x,y)$ *satisfying the sole assumption* $K(x,y) \leq C(xy)^{\lambda/2}$. *Let* $m \in \ell_{0+}$ *be fixed, and set, for* $k \geq 1$, $\mathbf{x}_k = \sqrt{C} m_k^{\lambda/2}$. *Then it is possible to couple the Marcus–Lushnikov process* $(M(m,t))_{t\geq 0}$ *with kernel* K *and the multiplicative coalescent* $(X(\mathbf{x},t))_{t\geq 0}$ *in such a way that a.s., for all* $t \geq 0$, $\|M(m,t)\|_\lambda \leq \|X(\mathbf{x},t)\|_2$.

Proof. We may assume by scaling that $C = 1$. We adopt the notation of Section 5, and consider in particular $Z(0)$ and a Poisson measure $O(\mathrm{d}t, \mathrm{d}(k,l), \mathrm{d}z)$ as in Remark 5.1.

Step 1: Recalling Remark 5.1, we consider the unique $\mathcal{P}$-valued solution $(\tilde{Z}(t))_{t\geq 0}$ of

$$\tilde{Z}(t) = Z(0) + \int_0^t \int_{k<l} \int_0^\infty \{\mathbf{c}_{k,l}(\tilde{Z}(s-)) - \tilde{Z}(s-)\} 1_{\{z \leq (m_k m_l)^{\lambda/2}\}} O(\mathrm{d}s, \mathrm{d}(k,l), \mathrm{d}z). \tag{6.2}$$

Due to Remark 5.1, the process $X(\mathbf{x}, t) := \mathcal{M}(\mathbf{x}, \tilde{Z}(t))$ is a multiplicative coalescent starting from $\mathbf{x}$. Indeed, setting $f(k,l,p) := (m_k m_l)^{\lambda/2}/\mu_k(\mathbf{x}, p)\mu_l(\mathbf{x}, p)$, we just have to check (5.7), which is immediate. We also consider the unique $\mathcal{P}$-valued solution $(Z(t))_{t\geq 0}$, see Remark 5.1, to

$$\begin{aligned} Z(t) = Z(0) &+ \int_0^t \int_{k<l} \int_0^\infty \{\mathbf{c}_{k,l}(Z(s-)) - Z(s-)\} \\ &\times 1_{\left\{z \leq (m_k m_l)^{\lambda/2} \frac{K[\mu_k(m, Z(s-)), \mu_l(m, Z(s-))]}{\mu_k(\mathbf{x}, Z(s-))\mu_l(\mathbf{x}, Z(s-))}\right\}} O(\mathrm{d}s, \mathrm{d}(k,l), \mathrm{d}z). \end{aligned} \tag{6.3}$$

Again, Remark 5.1 ensures that $M(m,t) := \mathcal{M}(m, Z(t))$ is a Marcus–Lushnikov process starting from m with coagulation kernel K.

Step 2. We now prove that a.s., for all $t \geq 0$, all $k \geq 1$, $Z_k(t) \subset \tilde{Z}_k(t)$. To do so, we consider the successive jump times $0 = T_0 < T_1 < T_2 < \cdots$ of Z, and the associated *marks* $((k_1, l_1), z_1)$, $((k_2, l_2), z_2), \ldots$ of the Poisson measure O.

We have $Z(t) = Z(T_i)$ for all $t \in [T_i, T_{i+1})$. On the other hand, for all $k \geq 1$, for all $0 \leq s \leq t$, clearly $\tilde{Z}_k(s) \subset \tilde{Z}_k(t)$. Hence it suffices to check that for all $i \geq 0$, all $k \geq 1$, $Z_k(T_i) \subset \tilde{Z}_k(T_i)$. We work by induction on i.

The result is obvious when $i = 0$. Assume that it holds for some $i - 1 \geq 0$. Observe that for $p \in \mathcal{P}$, $k \geq 1$, since $\lambda/2 \leq 1$,

$$\mu_k(m, p)^{\lambda/2} = \left(\sum_{i \in p_k} m_i\right)^{\lambda/2} \leq \sum_{i \in p_k} m_i^{\lambda/2} = \mu_k(\mathbf{x}, p). \tag{6.4}$$

Then, since T_i is a jump time of Z, we deduce that the corresponding mark $((k_i, l_i), z_i)$ satisfies, since $K(x,y) \leq (xy)^{\lambda/2}$,

$$\begin{aligned} z_i &\leq (m_{k_i} m_{l_i})^{\lambda/2} \frac{K[\mu_{k_i}(m, Z(T_{i-1})), \mu_{l_i}(m, Z(T_{i-1}))]}{\mu_{k_i}(\mathbf{x}, Z(T_{i-1}))\mu_{l_i}(\mathbf{x}, Z(T_{i-1}))} \\ &\leq (m_{k_i} m_{l_i})^{\lambda/2} \frac{[\mu_{k_i}(m, Z(T_{i-1}))\mu_{l_i}(m, Z(T_{i-1}))]^{\lambda/2}}{\mu_{k_i}(\mathbf{x}, Z(T_{i-1}))\mu_{l_i}(\mathbf{x}, Z(T_{i-1}))} \\ &\leq (m_{k_i} m_{l_i})^{\lambda/2}. \end{aligned} \tag{6.5}$$

We deduce from (6.5) that $\tilde{Z}(T_i) = \mathbf{c}_{k_i l_i}(\tilde{Z}(T_i-))$, while clearly, $Z(T_i) = \mathbf{c}_{k_i l_i}(Z(T_{i-1}))$. Since the inductive assumption ensures that for all $k \geq 1$, $Z_k(T_{i-1}) \subset \tilde{Z}_k(T_{i-1}) \subset \tilde{Z}_k(T_i-)$, we obviously deduce that $[\mathbf{c}_{k_i l_i}(Z(T_{i-1}))]_k \subset [\mathbf{c}_{k_i l_i}(\tilde{Z}(T_i-))]_k$ for all $k \geq 1$. The inductive proof is ended.

Step 3. We may now conclude: let $t \geq 0$ be fixed. Recall (5.6). Since $Z_k(t) \subset \tilde{Z}_k(t)$ for all $k \geq 1$, since $\lambda \geq 1$ and $\lambda/2 \leq 1$, using (6.4) again,

$$\|M(m,t)\|_\lambda = \sum_{k\geq 1} m_k \mu_k(m, Z(t))^{\lambda-1} \leq \sum_{k\geq 1} m_k \mu_k(m, \tilde{Z}(t))^{\lambda-1}$$

$$\leq \sum_{k\geq 1} \mathbf{x}_k^{2/\lambda} \mu_k(\mathbf{x}, \tilde{Z}(t))^{2(\lambda-1)/\lambda} = \sum_{k\geq 1} \mathbf{x}_k \mu_k(\mathbf{x}, \tilde{Z}(t)) \frac{\mathbf{x}_k^{2/\lambda-1}}{\mu_k(\mathbf{x}, \tilde{Z}(t))^{2/\lambda-1}}$$
$$\leq \sum_{k\geq 1} \mathbf{x}_k \mu_k(\mathbf{x}, \tilde{Z}(t)) = \|X(\mathbf{x}, t)\|_2. \tag{6.6}$$

This ends the proof. □

Proof of Theorem 6.2. We define $\mathcal{B} = \{\mathbf{x} \in \ell_{0+},\ \exists\, m \in \mathcal{A},\ \mathbf{x}_k = \sqrt{C} m_k^{\lambda/2} \forall\, k\}$, and its closure $\bar{\mathcal{B}}$ in ℓ_2. Using that $\lim_{i\to\infty} \sup_{m\in\mathcal{A}} \sum_{k\geq i} m_k^\lambda = 0$, we deduce that $\bar{\mathcal{B}}$ is a compact subset of ℓ_2. Theorem 6.1-(ii) implies that for each $t \geq 0$, the family of random variables $(X(\mathbf{x}, t))_{\mathbf{x}\in\bar{\mathcal{B}}}$ is tight in ℓ_2, so that in particular, the family of random variables $(\|X(\mathbf{x}, t)\|_2)_{\mathbf{x}\in\bar{\mathcal{B}}}$ is tight in $\mathbb{R}$. In other words, $\lim_{x\to\infty} \alpha(t, x) = 0$, where $\alpha(t, x) = \sup_{\mathbf{x}\in\bar{\mathcal{B}}} P[\|X(\mathbf{x}, t)\|_2 \geq x]$. Note that due to Lemma 6.3, for all $m \in \mathcal{A}$, a convenient coupling leads to $\|M(m, t)\|_\lambda \leq \|X(\mathbf{x}, t)\|_2$, where $\mathbf{x} := (\sqrt{C} m_1^{\lambda/2}, \sqrt{C} m_2^{\lambda/2}, \ldots) \in \mathcal{B}$, so that due to Theorem 6.1-(iii),

$$P\left[\sup_{[0,t]} \|M(m, s)\|_\lambda \geq x\right] \leq P\left[\sup_{[0,t]} \|X(\mathbf{x}, s)\|_2 \geq x\right]$$
$$= P\left[\|X(\mathbf{x}, t)\|_2 \geq x\right] \leq \alpha(t, x). \tag{6.7}$$

This concludes the proof. □

7. Finiteness when λ ≥ 2

We can of course not use, in this case, the result of Aldous. We will however follow the spirit of [1, Section 4]. Our aim in this section is to show the following result.

Theorem 7.1. *Let $\lambda \geq 2$ be fixed, and consider a coagulation kernel $K(x, y)$ satisfying the sole assumption $K(x, y) \leq Cxy(x\wedge y)^{\lambda-2}$. Consider a subset $\mathcal{A}$ of ℓ_{0+} such that $\sup_{m\in\mathcal{A}} \|m\|_\lambda < \infty$ and $\lim_{i\to\infty} \sup_{m\in\mathcal{A}} \sum_{k\geq i} m_k^\lambda = 0$. Then for each $t \geq 0$, $\lim_{x\to\infty} \alpha(t, x) = 0$, where*

$$\alpha(t, x) := \sup_{m\in\mathcal{A}} P\left[\sup_{s\in[0,t]} \|M(m, s)\|_\lambda \geq x\right], \tag{7.1}$$

where $(M(m, t))_{t\geq 0}$ stands for the Marcus–Lushnikov process starting from m with coagulation kernel K.

We first of all state and prove a comparison theorem.

Proposition 7.2. *Let $\lambda \geq 2$ and $C > 0$ be a constant. Consider the coagulation kernel $K_0(x, y) = Cxy(x \wedge y)^{\lambda-2}$, and another coagulation kernel K such that $K(x, y) \leq K_0(x, y)$ for all $x, y \geq 0$. Adopt the notation of Section 5. Consider, for $m \in \ell_{0+}$ the $\mathcal{P}$-valued processes $(V^m(t))_{t\geq 0}$ and $(Z^m(t))_{t\geq 0}$, unique solutions (recall Remark 5.1) of*

$$V^m(t) = Z(0) + \int_0^t \int_{k<l} \int_0^\infty \{\mathbf{c}_{k,l}(V^m(s-)) - V^m(s-)\}$$
$$\times 1_{\{z\leq Cm_k m_l[\mu_k(m,V^m(s-))\wedge\mu_l(m,V^m(s-))]^{\lambda-2}\}} O(\mathrm{d}s, \mathrm{d}(k,l), \mathrm{d}z), \tag{7.2}$$

$$Z^m(t) = Z(0) + \int_0^t \int_{k<l} \int_0^\infty \{\mathbf{c}_{k,l}(Z^m(s-)) - Z^m(s-)\} \times 1_{\left\{z \leq m_k m_l \frac{K[\mu_k(m,Z^m(s-)),\mu_l(m,Z^m(s-))]}{\mu_k(m,Z^m(s-))\mu_l(m,Z^m(s-))}\right\}} O(\mathrm{d}s, \mathrm{d}(k,l), \mathrm{d}z). \tag{7.3}$$

(i) *Then $(\mathcal{M}(m, V^m(t)))_{t\geq 0}$ (resp. $(\mathcal{M}(m, Z^m(t)))_{t\geq 0}$) is a Marcus–Lushnikov process starting from m with coagulation kernel K_0 (resp. K).*

(ii) *Furthermore, a.s., for all $m \in \ell_{0+}$, $t \geq 0$, $k \geq 1$, $Z_k^m(t) \subset V_k^m(t)$.*

(iii) *For all $t \geq 0$, $\|\mathcal{M}(m, Z^m(t))\|_\lambda \leq \|\mathcal{M}(m, V^m(t))\|_\lambda$ a.s.*

Proof. Point (i) is immediate from Remark 5.1. Setting, for $k < l$ and $p \in \mathcal{P}$, $f(k,l,p) := \frac{m_k m_l}{\mu_k(m,p)\mu_l(m,p)}$, we just have to prove that (5.7) holds. We have already checked it just after the statement of Remark 5.1.

To check point (ii), we follow the line of Step 2 of the proof of Theorem 6.2. We consider the successive jump times $0 = T_0 < T_1 < T_2 < \ldots$ of Z^m, and the associated *marks* $((k_1,l_1),z_1)$, $((k_2,l_2),z_2),\ldots$ of the Poisson measure O. We have $Z^m(t) = Z^m(T_i)$ for all $t \in [T_i, T_{i+1})$, and for all $k \geq 1$, for all $0 \leq s \leq t$, clearly $V_k^m(s) \subset V_k^m(t)$. Hence it suffices to check that for all $i \geq 0$, all $k \geq 1$, $Z_k^m(T_i) \subset V_k^m(T_i)$.

The result is obvious when $i = 0$. Assume that it holds for some $i - 1 \geq 0$. Then, since T_i is a jump time of Z^m, we deduce that the corresponding mark $((k_i,l_i),z_i)$ satisfies, since $K \leq K_0$,

$$\begin{aligned} z_i &\leq m_{k_i} m_{l_i} \frac{K[\mu_{k_i}(m, Z^m(T_{i-1})), \mu_{l_i}(m, Z^m(T_{i-1}))]}{\mu_{k_i}(m, Z^m(T_{i-1}))\mu_{l_i}(m, Z^m(T_{i-1}))} \\ &\leq C m_{k_i} m_{l_i} [\mu_{k_i}(m, Z^m(T_{i-1})) \wedge \mu_{l_i}(m, Z^m(T_{i-1}))]^{\lambda-2}. \end{aligned} \tag{7.4}$$

But by inductive assumption, for all $k \geq 1$, $Z_k^m(T_{i-1}) \subset V_k^m(T_{i-1}) \subset V_k^m(T_i-)$. Thus clearly, $\mu_k(m, Z^m(T_{i-1})) \leq \mu_k(m, V^m(T_i-))$ for all $k \geq 1$, so that we finally conclude that

$$z_i \leq C m_{k_i} m_{l_i} [\mu_{k_i}(m, V^m(T_i-)) \wedge \mu_{l_i}(m, V^m(T_i-))]^{\lambda-2}. \tag{7.5}$$

We deduce that $V^m(T_i) = \mathbf{c}_{k_i l_i}(V^m(T_i-))$. But $Z^m(T_i) = \mathbf{c}_{k_i l_i}(Z^m(T_{i-1}))$. By the inductive assumption, $Z_k^m(T_{i-1}) \subset V_k^m(T_{i-1}) \subset V_k^m(T_i-)$ for all $k \geq 1$. We conclude that $[\mathbf{c}_{k_i l_i}(Z^m(T_{i-1}))]_k \subset [\mathbf{c}_{k_i l_i}(V^m(T_i-))]_k$ for all $k \geq 1$. The inductive proof is ended.

We finally check Point (iii). Using that for all $k \geq 1$, $Z_k^m(t) \subset V_k^m(t)$, we immediately deduce that for all $k \geq 1$, $\mu_k(m, Z^m(t)) \leq \mu_k(m, V^m(t))$. Since $\lambda \geq 1$, (5.6) allows us to conclude. □

Notice that for given kernels $0 \leq K \leq K_0$, such comparison results do not in general hold. Proposition 7.2 could however be extended to any kernels satisfying $0 \leq K(x,y) \leq K_0(x,y)$ such that $x \mapsto K_0(x,y)/(xy)$ is non-decreasing for each y.

We next study the moment of order λ of the process $\mathcal{M}(m, V^m(t))$. We start with a lemma.

Lemma 7.3. *Adopt the notation and assumptions of Proposition 7.2. Set, for each $t \geq 0$, $m \in \ell_{0+}$, $S^m(t) := \|\mathcal{M}(m, V^m(t))\|_\lambda$. Then, setting $a := C2^{\lambda-1}\lambda$, we have for all $t \geq 0$,*

$$P(S^m(t) \leq 1) \geq 1 - (1 + at)\|m\|_\lambda. \tag{7.6}$$

Proof. Since $m \in \ell_{0+}$ and due to Proposition 7.2-(i), we have $S(t) := S^m(t) = \sum_{i\geq 1} M_i(t)^\lambda$, where $M(t)$ is a Marcus–Lushnikov process with initial condition m and coagulation kernel K_0. Then a simple computation shows that

$$E[1/S(t)] = 1/S(0) - \int_0^t \mathrm{d}s\, E[\Delta(s)], \tag{7.7}$$

where, since $M_j(t) \leq M_i(t)$, setting $\alpha_{ij}(t) := (M_i(t) + M_j(t))^\lambda - M_i(t)^\lambda - M_j(t)^\lambda$,

$$\Delta(t) = C \sum_{i<j} M_i(t) M_j(t)^{\lambda-1} \left(\frac{1}{S(t)} - \frac{1}{S(t) + \alpha_{ij}(t)} \right). \tag{7.8}$$

Note that $\alpha_{ij}(t) \leq (M_i(t) + M_j(t))^\lambda - M_i(t)^\lambda \leq \lambda (M_i(t) + M_j(t))^{\lambda-1} M_j(t)$ which is itself bounded by $2^{\lambda-1} \lambda M_i(t)^{\lambda-1} M_j(t)$, and hence

$$\Delta(t) \leq a \sum_{i<j} \frac{M_i(t)^\lambda M_j(t)^\lambda}{S(t)^2} \leq a. \tag{7.9}$$

We deduce that for $t \geq 0$, $E[1/S(t)] \geq 1/S(0) - at$. Next, we note that since $t \mapsto S(t)$ is non-decreasing,

$$\begin{aligned} E[1/S(t)] &= E[\mathbb{1}_{\{S(t) \leq 1\}}(1/S(t))] + E[\mathbb{1}_{\{S(t)>1\}}(1/S(t))] \\ &\leq (1/S(0)) P[S(t) \leq 1] + 1, \end{aligned} \tag{7.10}$$

so that finally,

$$P[S(t) \leq 1] \geq 1 - S(0)(at + 1). \tag{7.11}$$

This leads to (7.6), since $S(0) = \|m\|_\lambda$, because $V^m(0) = Z(0)$, so that $\mathcal{M}(m, V^m(0)) = m$. □

We now show that adding one particle at the beginning does not increase too much the moment of order λ.

Lemma 7.4. *Consider the same assumptions and notation as in Lemma 7.3. Let $m \in \ell_{0+}$, $x \geq m_1$, and set $m^x = (x, m_1, m_2, \dots) \in \ell_{0+}$. Then it is possible to couple the processes $(V^m(t))_{t \geq 0}$ and $(V^{m^x}(t))_{t \geq 0}$ (using two different Poisson measures) in such a way that a.s., for all $t \geq 0$,*

$$E\left[\left(S^{m^x}(t) \right)^{1/\lambda} | S^m(t) \right] \leq \left(S^m(t) \right)^{1/\lambda} + x \exp(C S^m(t) t), \tag{7.12}$$

the constant C being the one appearing in K_0. As a consequence, for all $y > 0$, all $A > 0$,

$$P\left[S^{m^x}(t) \leq A^\lambda \left(y^{1/\lambda} + x e^{Cyt} \right)^\lambda \right] \geq P\left[S^m(t) \leq y \right] (1 - 1/A). \tag{7.13}$$

Proof. *Step* 1. We consider a Poisson measure O as in Remark 5.1, and consider the process $(V^m(t))_{t \geq 0}$ defined in Proposition 7.2. We also introduce a second Poisson measure $M(\mathrm{d}s, \mathrm{d}i, \mathrm{d}z)$, independent of O, on $[0, \infty) \times \mathbb{N} \times [0, \infty)$, with intensity measure $\mathrm{d}t (\sum_{k \geq 1} \delta_k) \mathrm{d}z$.

Let $\bar{\mathcal{P}} = \{(p_n)_{n \geq 1},\ \forall n \geq 1,\ p_n = \emptyset \text{ or } n \in p_n \text{ and } \forall k \in p_n,\ p_k = p_n\}$. For $i \in \mathbb{N}$ and $p \in \bar{\mathcal{P}}$, we define $\mathbf{d}_i(p) \in \bar{\mathcal{P}}$ by

$$\forall k \in p_i, \quad [\mathbf{d}_i(p)]_k := \emptyset \quad \text{and} \quad \forall k \notin p_i, \quad [\mathbf{d}_i(p)]_k := p_k. \tag{7.14}$$

Let $(X(t), W(t))_{t\geq 0}$ be the unique solution, with values in $\mathbb{R}_+ \times \bar{\mathcal{P}}$, to

$$\begin{aligned} W(t) &= Z(0) + \int_0^t \int_{k<l} \int_0^\infty \{\mathbf{c}_{k,l}(W(s-)) - W(s-)\} \\ &\quad \times 1_{\{z \leq C m_k m_l [\mu_k(m, W(s-)) \wedge \mu_l(m, W(s-))]^{\lambda-2}\}} O(\mathrm{d}s, \mathrm{d}(k,l), \mathrm{d}z) \\ &\quad + \int_0^t \int_i \int_0^\infty \{\mathbf{d}_i(W(s-)) - W(s-)\} \\ &\quad \times 1_{\{z \leq C X(s-) m_i [X(s-) \wedge \mu_i(m, W(s-))]^{\lambda-2}\}} M(\mathrm{d}s, \mathrm{d}i, \mathrm{d}z), \\ X(t) &= x + \int_0^t \int_i \int_0^\infty \mu_i(m, W(s-)) \\ &\quad \times 1_{\{z \leq C X(s-) m_i [X(s-) \wedge \mu_i(m, W(s-))]^{\lambda-2}\}} M(\mathrm{d}s, \mathrm{d}i, \mathrm{d}z). \end{aligned} \tag{7.15}$$

Easy considerations show that the process $([X(t)]^\lambda + \|\mathcal{M}(m, W(t))\|_\lambda)_{t\geq 0}$ is a version of $(S^{m^x}(t))_{t\geq 0}$, admitting the natural convention that if $p_k = \emptyset$, $\mu_k(m, p) = 0$. Indeed, W represents the particles which have not coalesced with x, and X stands for the mass of the particle containing x.

Step 2. We now prove that a.s., for all $t \geq 0$, all $k \geq 1$, $W_k(t) \subset V_k^m(t)$.

Consider the successive jump times $0 = T_0 < T_1 < \ldots$ of W. It suffices, as usual, to show that for each $i \geq 0$, each $k \geq 1$, $W_k(T_i) \subset V_k^m(T_i)$. This is obvious for $i = 0$, and we assume that it holds for some $i - 1$. Then, if T_i is a jump of M, clearly we have $V^m(T_i) = V^m(T_i-)$, while for some k_i, $W(T_i) = \mathbf{d}_{k_i}[W(T_{i-1})]$. This implies, using the inductive assumption, that for all $k \geq 1$, $W_k(T_i) \subset V_k^m(T_i)$.

Next, if T_i is a time of jump of O, we denote by $((k_i, l_i), z_i)$ the associated mark of O, and conclude, using the same arguments as usual, that $W(T_i) = \mathbf{c}_{k_i l_i}(W(T_{i-1}))$ and $V^m(T_i) = \mathbf{c}_{k_i l_i}(V^m(T_i-))$. Since by inductive assumption, $W_k(T_{i-1}) \subset V_k^m(T_{i-1}) \subset V_k^m(T_i-)$ for all $k \geq 1$, the conclusion follows.

Step 3. Due to Steps 1 and 2, we have a.s., for all $t \geq 0$,

$$S^{m^x}(t) \leq S^m(t) + [X(t)]^\lambda. \tag{7.16}$$

Indeed, recalling (5.6),

$$\begin{aligned} \|\mathcal{M}(m, W(t))\|_\lambda &= \sum_{i\geq 1} m_i [\mu_i(m, W(t))]^{\lambda-1} \\ &\leq \sum_{i\geq 1} m_i [\mu_i(m, V^m(t))]^{\lambda-1} = S^m(t). \end{aligned} \tag{7.17}$$

Step 4. We now prove (7.12). Consider the σ-field $\mathcal{G} := \sigma(O)$. Since $W_k(s-) \subset V_k^m(s-)$ for all $k \geq 1$, we obtain

$$\begin{aligned} X(t) &\leq x + \int_0^t \int_i \int_0^\infty \mu_i(m, V^m(s-)) \\ &\quad \times 1_{\{z \leq C X(s-) m_i [X(s-) \wedge \mu_i(m, V^m(s-))]^{\lambda-2}\}} M(\mathrm{d}s, \mathrm{d}i, \mathrm{d}z). \end{aligned} \tag{7.18}$$

But $(V^m(t))_{t\geq 0}$ is $\mathcal{G}$-measurable, while the Poisson measure M is independent of $\mathcal{G}$. Hence,

$$E^{\mathcal{G}}[X(t)] \leq x + C \int_0^t \mathrm{d}s \sum_{i\geq 1} m_i \mu_i(m, V^m(s)) E^{\mathcal{G}}\left[X(s)[X(s) \wedge \mu_i(m, V^m(s))]^{\lambda-2}\right]$$

$$\leq x + C\int_0^t \mathrm{d}s\, E^{\mathcal{G}}[X(s)] \sum_{i\geq 1} m_i \mu_i(m, V^m(s))^{\lambda-1}$$
$$\leq x + CS^m(t)\int_0^t \mathrm{d}s\, E^{\mathcal{G}}[X(s)], \tag{7.19}$$

where we used (5.6) and the fact that $t \mapsto S^m(t)$ is a.s. non-decreasing to obtain the last inequality. The Gronwall Lemma allows us to conclude that $E^{\mathcal{G}}[X(t)] \leq x\exp(CtS^m(t))$. Hence, recalling (7.16), and noting that $1/\lambda < 1$,

$$E\left[\left(S^{m^x}(t)\right)^{1/\lambda} |\mathcal{G}\right] \leq \left(S^m(t)\right)^{1/\lambda} + x\exp(CS^m(t)t), \tag{7.20}$$

which concludes the proof of (7.12) since $S^m(t)$ is $\mathcal{G}$-measurable.

Step 5. It is straightforward to deduce (7.13): for $y > 0$, $A > 0$,

$$P\left[S^{m^x}(t) \leq A^\lambda \left(y^{1/\lambda} + x\mathrm{e}^{Cyt}\right)^\lambda\right]$$
$$\geq E\left[1_{\{S^m(t)\leq y\}} P\left\{(S^{m^x}(t))^{1/\lambda} \leq A\left((S^m(t))^{1/\lambda} + x\mathrm{e}^{CS^m(t)t}\right)\Big| S^m(t)\right\}\right]$$
$$\geq E\left[1_{\{S^m(t)\leq y\}}\left(1 - \frac{E\left\{(S^{m^x}(t))^{1/\lambda}|S^m(t)\right\}}{A\left((S^m(t))^{1/\lambda} + x\mathrm{e}^{CS^m(t)t}\right)}\right)\right]$$
$$\geq P[S^m(t) \leq y](1 - 1/A). \tag{7.21}$$

We used (7.12) to obtain the last inequality. □

Proof of Theorem 7.1. Due to Proposition 7.2, it suffices to prove the result when $K = K_0$. Consider thus a subset $\mathcal{A}$ of ℓ_{0+} such that $c := \sup_{m\in\mathcal{A}} \|m\|_\lambda < \infty$ and $\varepsilon_k := \sup_{m\in\mathcal{A}} \sum_{i\geq k} m_i^\lambda$ tends to 0 as k tends to infinity. Observe at once that for all $m \in \mathcal{A}$, all $k \geq 1$, $m_k \leq b := c^{1/\lambda}$.

Using the notation of Lemma 7.3, and since $t \mapsto S^m(t)$ is non-decreasing, we just have to prove that for all $t \geq 0$,

$$\lim_{y\to\infty} \sup_{m\in\mathcal{A}} P[S^m(t) \geq y] = 0. \tag{7.22}$$

Let $t \geq 0$ and $\eta > 0$ be fixed. We want to show that there exists y such that for all $m \in \mathcal{A}$, $P[S^m(t) \leq y] \geq 1 - \eta$. First of all, we may find k large enough, such that $\varepsilon_k \leq \eta/2(1 + at)$. Then for all $m \in \mathcal{A}$, setting $m^{[k]} := (m_k, m_{k+1}, \ldots)$, we have $\|m^{[k]}\|_\lambda \leq \eta/2(1 + at)$. Hence Lemma 7.3 ensures that $P[S^{m^{[k]}} \leq 1] \geq 1 - \eta/2$. Since, with the notation of Lemma 7.4, $m^{[k-1]} = (m^{[k]})^{m_{k-1}}$, we obtain, due to (7.13) that setting $y_1 := A_1^\lambda(1 + be^{Ct})^\lambda$ for some A_1 such that $(1 - \eta/2)(1 - 1/A_1) \geq 1 - 3\eta/4$,

$$P[S^{m^{[k-1]}} \leq y_1] \geq (1 - \eta/2)(1 - 1/A_1) \geq 1 - 3\eta/4. \tag{7.23}$$

The same argument yields that setting now $y_2 := A_2^\lambda(y_1^{1/\lambda} + be^{Cy_1t})^\lambda$ for some A_2 such that $(1 - 3\eta/4)(1 - 1/A_2) \geq 1 - 7\eta/8$,

$$P[S^{m^{[k-2]}} \leq y_2] \geq (1 - 3\eta/4)(1 - 1/A_2) \geq 1 - 7\eta/8. \tag{7.24}$$

A backward induction allows us to conclude that we may find $y = y_{k-1} \in (0, \infty)$ such that for all $m \in \mathcal{A}$, $P[S^m(t) \leq y] \geq 1 - \eta$. This concludes the proof. □

8. Existence and uniqueness for (*SDE*)

We may now prove the existence of a solution to (*SDE*).

Theorem 8.1. *Let $\lambda \in \mathbb{R} \setminus \{0\}$ be fixed and assume $A(\lambda)$. Consider a Poisson measure N as in Definition 3.1. For each $m \in \ell_\lambda$, there exists a unique solution $(M(m,t))_{t\geq 0}$ to $(SDE)(\lambda, m, K, N)$.*

Proof. We consider a Poisson measure N as in Definition 3.1, and we fix $m \in \ell_\lambda$. First of all, the uniqueness assertion follows directly from Proposition 4.1. We just have to show the existence of a solution.

If $\lambda < 0$, we consider $m^n = (m_1, \ldots, m_n, \infty, \ldots) \in \ell_{0-}$, while if $\lambda > 0$, we set $m^n = (m_1, \ldots, m_n, 0, \ldots) \in \ell_{0+}$. We denote by $M^n(t) = M(m^n, t)$ the unique solution to $(SDE)(\lambda, m^n, K, N)$, obtained in Remark 3.2: for each $n \geq 1$, all $k \geq 1$, $t \geq 0$,

$$[M_k^n(t)]^\lambda = [m_k^n]^\lambda + \int_0^t \int_{i<j} \int_0^\infty [[c_{ij}(M^n(s-))]_k^\lambda - [M_k^n(s-)]^\lambda] \times 1_{\{z \leq K(M_i^n(s-), M_j^n(s-))\}} N(\mathrm{d}s, \mathrm{d}(i,j), \mathrm{d}z). \tag{8.1}$$

We divide the proof into four cases.

Case 1: $\lambda < 0$. Set $a := \|m\|_\lambda = \sup_{n\geq 1} \sup_{t\geq 0} \|M^n(t)\|_\lambda$, and $\varepsilon := m_1 = \inf_{n\geq 1} \inf_{t\geq 0} M_n(t)$. Due to Proposition 4.1, we have for all $t \geq 0$,

$$E\left[\sup_{s\in[0,t]} d_\lambda(M^n(s), M^{n+1}(s))\right] \leq \mathrm{e}^{16aC_\varepsilon t} d_\lambda(m^n, m^{n+1}). \tag{8.2}$$

Hence there exists a $(\mathcal{F}_t)_{t\geq 0}$-adapted and $\mathbb{D}([0,\infty), \ell_\lambda)$-valued process $(M(t))_{t\geq 0}$ such that for all $t \geq 0$,

$$\lim_n E\left[\sup_{s\in[0,t]} d_\lambda(M^n(s), M(s))\right] = 0. \tag{8.3}$$

Furthermore, we have a.s., for all $t \geq 0$, $M_1(t) \geq \varepsilon$ and $\|M(t)\|_\lambda \leq a$. To pass to the limit in (8.1), it suffices to show that $\lim_n \Delta_n(t) = 0$, where

$$\begin{aligned}\Delta_n(t) &= E\Bigg[\int_0^t \int_{i<j} \int_0^\infty N(\mathrm{d}s, \mathrm{d}(i,j), \mathrm{d}z) \\ &\quad \times \sum_k \Big| \left([c_{ij}(M(s-))]_k^\lambda - [M_k(s-)]^\lambda\right) 1_{\{z\leq K(M_i(s-)), M_j(s-)\}} \\ &\quad - \left([c_{ij}(M^n(s-))]_k^\lambda - [M_k^n(s-)]^\lambda\right) 1_{\{z\leq K(M_i^n(s-)), (M_j^n(s-))\}} \Big|\Bigg] \\ &\leq A_n(t) + B_n(t),\end{aligned} \tag{8.4}$$

where $A_n(t) = \sum_{i<j} A_n^{ij}(t)$, with

$$\begin{aligned}A_n^{ij}(t) &= E\Bigg[\int_0^t \mathrm{d}s\, K(M_i(s), M_j(s)) \sum_k \Big|([c_{ij}(M(s))]_k^\lambda \\ &\quad - [M_k(s)]^\lambda) - \left([c_{ij}(M^n(s))]_k^\lambda - [M_k^n(s)]^\lambda\right) \Big|\Bigg]\end{aligned} \tag{8.5}$$

and

$$B_n(t) = E\left[\int_0^t \mathrm{d}s \sum_{i<j} |K(M_i(s), M_j(s)) - K(M_i^n(s), M_j^n(s))| \sum_k |[c_{ij}(M^n(s))]_k^\lambda - [M_k^n(s)]^\lambda|\right]. \tag{8.6}$$

Using (A.4) and then (2.5), we obtain

$$\begin{aligned} B_n(t) &\le \int_0^t \mathrm{d}s\, E\left[\sum_{i<j} |K(M_i(s), M_j(s)) - K(M_i^n(s), M_j^n(s))| d_\lambda(c_{ij}(M^n(s)), M^n(s))\right] \\ &\le 4C_\varepsilon \int_0^t \mathrm{d}s\, E\left[(\|M(s)\|_\lambda + \|M^n(s)\|_\lambda) d_\lambda(M^n(s), M(s))\right] \\ &\le 8aC_\varepsilon \int_0^t \mathrm{d}s\, E\left[d_\lambda(M^n(s), M(s))\right], \end{aligned} \tag{8.7}$$

which tends to 0 as $n \to \infty$ due to (8.3). In order to show that $A_n(t)$ tends to 0, it is sufficient to show that

(a) for each $1 \le i < j$, $A_n^{ij}(t)$ tends to 0 as $n \to \infty$,

(b) $\lim_{k\to\infty} \limsup_{n\to\infty} \sum_{i+j\ge k} A_n^{ij}(t) = 0$.

For each $i < j$, using (2.4) and (A.5), we have

$$\begin{aligned} A_n^{ij}(t) &\le C_\varepsilon \int_0^t \mathrm{d}s\, E\left[(M_i(s) + M_j(s))^\lambda \right. \\ &\qquad \left. \times \left(d_\lambda(c_{ij}(M(s)), c_{ij}(M^n(s))) + d_\lambda(M(s), M^n(s))\right)\right] \\ &\le 2C_\varepsilon \varepsilon^\lambda \int_0^t \mathrm{d}s\, E\left[d_\lambda(M^n(s), M(s))\right], \end{aligned} \tag{8.8}$$

which tends to 0. This implies (a). On the other hand, by (2.4) and (A.4),

$$\begin{aligned} A_n^{ij}(t) &\le C_\varepsilon \int_0^t \mathrm{d}s\, E\left[(M_i(s) + M_j(s))^\lambda (d_\lambda(c_{ij}(M(s)), M(s)) \right. \\ &\qquad \left. + \; d_\lambda(c_{ij}(M^n(s)), M^n(s)))\right] \\ &\le 2C_\varepsilon \int_0^t \mathrm{d}s\, E\left[M_j(s)^\lambda \left(M_i(s)^\lambda + M_i^n(s)^\lambda\right)\right]. \end{aligned} \tag{8.9}$$

Thus,

$$\limsup \sum_{i+j\ge k} A_n^{ij}(t) \le 4C_\varepsilon \int_0^t \mathrm{d}s\, E\left[\sum_{i+j\ge k} M_j(s)^\lambda M_i(s)^\lambda\right], \tag{8.10}$$

which tends to 0 as k tends to infinity by the Lebesgue Theorem, since $(M(s))_{s\ge 0}$ belongs a.s. to $\mathbb{D}([0,\infty), \ell_\lambda)$ and satisfies $\|M(s)\|_\lambda \le a$ a.s. for all $s \ge 0$. This concludes the proof when $\lambda < 0$.

Case 2: $\lambda \in (0, 1]$. This case is handled as Case 1, noting that $a := \|m\|_\lambda = \sup_{n\ge 1}\sup_{t\ge 0}\|M^n(t)\|_\lambda$, and $b := \|m\|_1 = \sup_{n\ge 1}\sup_{t\ge 0}\|M^n(t)\|_1$, and that for all $k \ge 1$, $n \ge 1$, $t \ge 0$, $M_k^n(t) \le b$ a.s.

Case 3: $\lambda \in (1, 2]$. This case is more complicated. Consider, for each $n \geq 1$, $x \in (0, \infty)$, the stopping time $\tau_x^n := \inf\{t \geq 0, \|M^n(t)\|_\lambda \geq x\}$. Using Theorem 6.2 and since $m \in \ell_\lambda$, we know that setting, for $x \in (0, \infty)$, $t \geq 0$,

$$\alpha(t, x) := \sup_{n \geq 1} P[\tau_x^n \leq t], \qquad \text{we have } \lim_{x \to \infty} \alpha(t, x) = 0. \tag{8.11}$$

Furthermore, we know from Proposition 4.1 that for all $n \geq 1$, all $x \in (0, \infty)$, all $T \geq 0$,

$$E\left[\sup_{[0, T \wedge \tau_x^n \wedge \tau_x^{n+1})} d_\lambda(M^n(s), M^{n+1}(s))\right] \leq d_\lambda(m^n, m^{n+1}) \mathrm{e}^{CxT}. \tag{8.12}$$

It is not difficult to deduce from (8.11) and (8.12), and the fact that $(m^n)_{n \geq 1}$ is a Cauchy sequence for d_λ, that for all $\varepsilon > 0$, $T > 0$, we may find $n_\varepsilon > 0$ such that for all $p, q \geq n_\varepsilon$,

$$P\left[\sup_{[0,T]} d_\lambda(M^p(s), M^q(s)) \geq \varepsilon\right] \leq \varepsilon. \tag{8.13}$$

Indeed, for all $x \in (0, \infty)$,

$$\begin{aligned} P\left[\sup_{[0,T]} d_\lambda(M^p(s), M^q(s)) \geq \varepsilon\right] &\leq P[\tau_x^p \leq T] + P[\tau_x^q \leq T] \\ &\quad + \frac{1}{\varepsilon} E\left[\sup_{[0, T \wedge \tau_x^p \wedge \tau_x^q)} d_\lambda(M^p(s), M^q(s))\right] \\ &\leq 2\alpha(T, x) + \frac{1}{\varepsilon} d_\lambda(m^p, m^q) \mathrm{e}^{CxT}. \end{aligned} \tag{8.14}$$

Choosing first x large enough so that $\alpha(T, x) \leq \varepsilon/4$ and then n_ε large enough, in such a way that for all $p, q \geq n_\varepsilon$, $d_\lambda(m^p, m^q) \leq (\varepsilon^2/2)\mathrm{e}^{-CxT}$, we conclude that (8.13) holds.

We deduce from (8.13) that the sequence of processes $(M^n(t))_{t \geq 0}$ is Cauchy in probability in $\mathbb{D}([0, \infty), \ell_\lambda)$, endowed with the uniform norm in time on compact intervals. We thus may find a subsequence (not relabelled) and a $(\mathcal{F}_t)_{t \geq 0}$-adapted process $(M(t))_{t \geq 0}$ belonging a.s. to $\mathbb{D}([0, \infty), \ell_\lambda)$ such that for all $T > 0$,

$$\text{a.s.,} \ \lim_n \sup_{[0,T]} d_\lambda(M^n(s), M(s)) = 0. \tag{8.15}$$

Set now $\tau_x := \inf\{t \geq 0, \|M(t)\|_\lambda \geq x\}$. Due to the Lebesgue Theorem,

$$\lim_n E\left[\sup_{[0, T \wedge \tau_x^n \wedge \tau_x)} d_\lambda(M^n(s), M(s))\right] = 0. \tag{8.16}$$

We want to show that $(M(t))_{t \geq 0}$ solves $(SDE)(\lambda, K, m, N)$. To do this, we want to pass to the limit in (3.1). It suffices to check that $\lim_n \Delta_n(t, x) = 0$, where $\Delta_n(t, x) \leq \sum_{i<j} A_{ij}^n(t, x) + B_n(t, x)$ are defined as in (8.4)–(8.6), replacing all the integrals $\int_0^t$ by $\int_0^{t \wedge \tau_x^n \wedge \tau_x}$.

This will suffice since due to (8.15), for all $x \in (0, \infty)$, a.s., for n large enough, $\tau_x^n \geq \tau_{x/2}$. Thus M will solve $(SDE)(\lambda, m, K, N)$ on the time interval $[0, \tau_{x/2})$ for all $x > 0$, and thus on $[0, \infty)$, since a.s., $\lim_{x \to \infty} \tau_x = \infty$, because $M \in \mathbb{D}([0, \infty), \ell_\lambda)$ a.s. First we obtain, using (2.9) and (A.10)

$$\begin{aligned}
B_n(t,x) &\le CE\left[\int_0^{t\wedge\tau_x^n\wedge\tau_x} \mathrm{d}s \sum_{i<j} |K(M_i(s-),M_j(s-)) - K(M_i^n(s-),M_j^n(s-))| \right.\\
&\qquad \left. \times\; M_j^n(s-)M_i^n(s-)^{\lambda-1}\right]\\
&\le CE\left[\int_0^{t\wedge\tau_x^n\wedge\tau_x} \mathrm{d}s(\|M(s-)\|_\lambda + \|M^n(s-)\|_\lambda) d_\lambda(M^n(s-),M(s-))\right]\\
&\le CxtE\left[\sup_{[0,t\wedge\tau_x\wedge\tau_x^n)} d_\lambda(M^n(s),\tilde{M}(s))\right],
\end{aligned} \tag{8.17}$$

which tends to 0 as $n\to\infty$ due to (8.16). To show that $A_n(t) = \sum_{i<j} A_{ij}^n(t,x)$ tends to 0, we check that

(a) for each $1 \le i < j$, $A_n^{ij}(t,x)$ tends to 0 as $n\to\infty$,
(b) $\lim_{k\to\infty} \limsup_{n\to\infty} \sum_{i+j\ge k} A_n^{ij}(t,x) = 0$.

First, for each $i<j$, using (2.8) and (A.11)

$$\begin{aligned}
A_n^{ij}(t,x) &\le E\left[\int_0^{t\wedge\tau_x^n\wedge\tau_x} \mathrm{d}s K(M_i(s),M_j(s)) \right.\\
&\qquad \left. \times\; \left\{d_\lambda(c_{ij}(M(s)),c_{ij}(M^n(s))) + d_\lambda(M(s),M^n(s))\right\}\right]\\
&\le CE\left[\int_0^{t\wedge\tau_x^n\wedge\tau_x} \mathrm{d}s K(M_i(s),M_j(s))\{2d_\lambda(M(s),M^n(s)) \right.\\
&\qquad \left. +\; |d_\lambda(c_{ij}(M(s)),c_{ij}(M^n(s))) - d_\lambda(M(s),M^n(s))|\}\right]\\
&\le CE\left[\int_0^{t\wedge\tau_x^n\wedge\tau_x} \mathrm{d}s \left(\|M^n(s)\|_\lambda + \|M(s)\|_\lambda\right) d_\lambda(M(s),M^n(s))\right]\\
&\le CxtE\left[\sup_{[0,t\wedge\tau_x\wedge\tau_x^n)} d_\lambda(M^n(s),\tilde{M}(s))\right],
\end{aligned} \tag{8.18}$$

which tends to 0 as $n\to\infty$. To show (b), note that due to (2.8) and (A.10),

$$\begin{aligned}
A_n^{ij}(t,x) &\le E\left[\int_0^{t\wedge\tau_x^n\wedge\tau_x} \mathrm{d}s K(M_i(s),M_j(s)) \right.\\
&\qquad \left. \times\; \left(d_\lambda(c_{ij}(M(s)),M(s)) + d_\lambda(c_{ij}(M^n(s)),M^n(s))\right)\right]\\
&\le CE\left[\int_0^{t\wedge\tau_x^n\wedge\tau_x} \mathrm{d}s[M_i(s)M_j(s)]^{\lambda/2}\left\{M_j(s)M_i(s)^{\lambda-1} + M_j^n(s)M_i^n(s)^{\lambda-1}\right\}\right]
\end{aligned}$$

$$\leq CE\left[\int_0^{t\wedge\tau_x^n\wedge\tau_x} \mathrm{d}s[M_i(s)M_j(s)]^{\lambda/2} \right.$$
$$\left.\times \left\{(M_j(s)M_i(s))^{\lambda/2} + (M_j^n(s)M_i^n(s))^{\lambda/2}\right\}\right]. \tag{8.19}$$

Thus,

$$\limsup_n \sum_{i+j>k} A_n^{ij}(t,x) \leq CE\left[\int_0^{t\wedge\tau_x} \mathrm{d}s \sum_{i+j>k}[M_i(s-)M_j(s-)]^{\lambda}\right], \tag{8.20}$$

which tends to 0 as k tends to infinity as usual by the Lebesgue Theorem, since we work on the time interval $[0,\tau_x)$.

Case 4: $\lambda \geq 2$. It is handled as Case 3, making use of Theorem 7.1 and (2.10) instead of Theorem 6.2 and (2.8). □

9. Conclusion

It remains to conclude the proof of Theorem 2.2 and Proposition 2.3.

We start with some boundedness of the operator $\mathcal{L}$.

Lemma 9.1. *Let $\lambda \in \mathbb{R}\setminus\{0\}$, and assume $A(\lambda)$. Let $\Phi : \ell_\lambda \mapsto \mathbb{R}$ satisfy, for all $m, \tilde{m} \in \ell_\lambda$, $|\Phi(m)| \leq a$ and $|\Phi(m) - \Phi(\tilde{m})| \leq a d_\lambda(m,\tilde{m})$. Recall (2.3). Then $m \mapsto \mathcal{L}\Phi(m)$ is bounded on $\{m \in \ell_\lambda, \|m\|_\lambda \leq c\}$ for each $c > 0$.*

Proof. This lemma is a straightforward consequence of $A(\lambda)$ and Lemma A.2. Let us for example study the case $\lambda < 0$. Let $c > 0$ be fixed, and set $\varepsilon := 1/c^{1/|\lambda|}$. Notice that if $\|m\|_\lambda \leq c$, then for all $k \geq 1$, $m_k \geq \varepsilon$. Due to (2.4) and (A.4), we have for all $m \in \ell_\lambda$ such that $\|m\|_\lambda \leq c$,

$$|\mathcal{L}\Phi(m)| \leq C_\varepsilon \sum_{i<j}(m_i+m_j)^\lambda |\Phi(c_{ij}(m)) - \Phi(m)| \leq aC_\varepsilon \sum_{i<j} m_j^\lambda d_\lambda(m, c_{ij}(m))$$
$$\leq 2aC_\varepsilon \sum_{i<j} m_i^\lambda m_j^\lambda \leq 2aC_\varepsilon c^2. \tag{9.1}$$

The boundedness of $\mathcal{L}\Phi$ on $\{m \in \ell_\lambda,\ m_1 \geq \varepsilon,\ \|m\|_\lambda \leq c\}$ is proved. □

Proof of Theorem 2.2. Let $\lambda \in \mathbb{R}\setminus\{0\}$ and a coagulation kernel K satisfying $A(\lambda)$ be fixed. Consider a Poisson measure N as in Definition 3.1. For each $m \in \ell_\lambda$, denote by $(M(m,t))_{t\geq 0}$ the unique solution to $(SDE)(\lambda, m, K, N)$, built in Theorem 8.1. It is a strong Markov Process, since it solves a time-homogeneous Poisson-driven S.D.E. for which pathwise uniqueness holds. To check points (i) and (ii), we consider the different cases separately.

Case 1: $\lambda < 0$. This case is almost obvious using Remark 3.2 and Proposition 4.1. Consider $m^n, m \in \ell_\lambda$ such that $\lim d_\lambda(m^n, m) = 0$. Then clearly, $a := \|m\|_\lambda + \sup_{n\geq 1}\|m^n\|_\lambda < \infty$, while $\varepsilon := m_1 \wedge \inf_n m_1^n > 0$. We thus conclude, due to Proposition 4.1, that for all $T > 0$, $E[\sup_{[0,T]} d_\lambda(M(m^n,t), M(m,t))]$ tends to 0 as $n \to \infty$, which concludes the proof.

Case 2: $\lambda \in (0,1]$. It is handled as Case 1, noting that if $m^n, m \in \ell_\lambda$ such that $\lim d_\lambda(m^n, m) = 0$, then $a := \sup_n \|m^n\|_\lambda < \infty$ and $b := \sup_n \|m^n\|_1 < \infty$.

Case 3: $\lambda \in (1,2]$. Denote by $\tau(m,x) = \inf\{t \geq 0,\ \|M(m,t)\|_\lambda \geq x\}$. Consider a sequence $m^n \in \ell_{0+}$ such that $\lim_n d_\lambda(m^n, m) = 0$. First of all notice that due to Theorem 6.2, we have for all $t > 0$,

$$\lim_{x\to\infty} \alpha(t,x) = 0 \quad \text{where } \alpha(t,x) := \sup_n P[\tau(m^n, x) \le t]. \tag{9.2}$$

It is easily deduced from Proposition 4.1 that for all $T \ge 0$, $\varepsilon > 0$,

$$\lim_n P\left[\sup_{[0,T]} d_\lambda(M(m^n,t), M(m,t)) > \varepsilon\right] = 0. \tag{9.3}$$

Indeed, it suffices to note that for all $x \in (0,\infty)$,

$$P\left[\sup_{[0,T]} d_\lambda(M(m^n,t), M(m,t)) > \varepsilon\right] \le \alpha(T,x) + P[\tau(m,x) \le T] + \frac{1}{\varepsilon} \mathrm{e}^{CTx} d_\lambda(m^n, m), \tag{9.4}$$

to make first n and then x tends to infinity. Hence (i) holds.

Next, we extend Theorem 6.2 in the following way: for any subset $\mathcal{A}$ of ℓ_λ (not only ℓ_{0+}) such that $\sup_{m\in\mathcal{A}} \|m\|_\lambda < \infty$ and $\lim_{i\to\infty} \sup_{m\in\mathcal{A}} \sum_{k\ge i} m_k^\lambda = 0$ and for any $t \ge 0$, $\lim_{x\to\infty} \alpha(t,x) = 0$, where

$$\alpha(t,x) := \sup_{m\in\mathcal{A}} P[\tau(m,x) \le t]. \tag{9.5}$$

Indeed, for each $m \in \mathcal{A}$, consider $m^n = (m_1, \ldots, m_n, 0, \ldots)$, and denote by $\mathcal{A}_0 := \{m^n,\ n \ge 1, m \in \mathcal{A}\}$. Then due to Theorem 6.2, we know that $\lim_{x\to\infty} \beta(t,x) = 0$, where $\beta(t,x) := \sup_{m\in\mathcal{A}_0} P[\tau(m,x) \le t]$. Using point (i) (see (9.3)), we easily deduce that for $m \in \ell_\lambda$, $P[\tau(m,x) \le t] \le \limsup_n P[\tau(m^n, x/2) \le t]$. Hence $\alpha(t,x) \le \beta(t, x/2)$, and $\lim_{x\to\infty} \alpha(t,x) = 0$.

Using the uniform bound (9.5), we may prove point (ii) exactly as point (i).

Case 4: $\lambda > 2$. It is the same proof as when $\lambda \in (1,2]$, making use of Theorem 7.1 instead of Theorem 6.2.

Point (iii) is straightforward since $(M(m,t))_{t\ge0}$ solves $(SDE)(\lambda, K, m, N)$. In the case where $\lambda < 0$ or $\lambda \in (0,1]$, use that $\mathcal{L}\Phi(M(m,s))$ is uniformly bounded since $\sup_{t\ge0} \|M(m,t)\|_\lambda = \|m\|_\lambda$ and due to Lemma 9.1. If $\lambda > 1$, use the sequence of stopping times $\tau(m, x_n)$ (with $x_n = n$ for all $n \ge 1$) and that $\mathcal{L}\Phi(M(m,s))_{s\in[0,\tau(m,x_n))}$ is uniformly bounded due to Lemma 9.1. □

We finally show that no infinite particles may appear, under our assumptions, when $\lambda < 0$. Again, we have not found any easy moment estimate which would allow us to conclude. We thus have to work more precisely, using again the *complete* description defined in Section 5.

Proof of Theorem 2.3. We split the proof into several parts. We fix $m \in \ell_\lambda \setminus \ell_{0-}$.

Step 1. In spirit of the de la Vallée Poussin Theorem and since $m \in \ell_\lambda \setminus \ell_{0-}$, we may find a non-decreasing function $\Lambda : [m_1, \infty) \mapsto \mathbb{R}_+$ such that $\lim_{x\to\infty} \Lambda(x) = \infty$ and $\sum_{k\ge1} m_k^\lambda \Lambda(m_k) < \infty$. Furthermore, Λ can be chosen to be continuous and in such a way that $\Lambda(m_1) = 1$, and that $x \mapsto x^{-1}\Lambda(x)$ and $x \mapsto x^\lambda \Lambda(x)$ are non-increasing. This is done in the following way: there exists a non-decreasing C^1 function $\varphi : [1,\infty) \mapsto [m_1, \infty)$ such that $\varphi(1) = m_1$, $\varphi(\infty) = \infty$, satisfying

$$\sum_k m_k^\lambda 1_{\{m_k \ge \varphi(x)\}} \le \frac{C}{1+x^2} \quad \text{for all } x > 0. \tag{9.6}$$

If $|\lambda| \geq 1$, we set $\psi(x) := x \sup_{u\in[1,x]} u^{-1}\varphi(u)$, while if $|\lambda| < 1$, we put $\psi(x) := x^{1/|\lambda|} \sup_{u\in[1,x]} u^{-1/|\lambda|}\varphi(u)$. In any case, ψ is non-decreasing, greater than φ, and still satisfies $\psi(1) = m_1$. Then one may show that the function $\Lambda(y) = \psi^{-1}(y) = 1 + \int_1^\infty 1\!\!1_{\{\psi(x)\leq y\}}\mathrm{d}x$ fulfills the required conditions.

Define also, for all $A > 0$, $\Lambda_A(x) := \Lambda(x \wedge A)$. Then $x \mapsto x^\lambda \Lambda_A(x)$ is non-increasing, and for all $x, y \in (0, \infty)$, $\Lambda_A(x + y) \leq \Lambda_A(x) + \Lambda_A(y)$.

Step 2. Notice that, since $x \mapsto x^\lambda \Lambda_A(x)$ is non-increasing, we easily deduce that for $i < j$, $\tilde{m} \in \ell_\lambda$, $\sum_{p\geq 1} \Lambda_A([c_{ij}(\tilde{m})]_p)[c_{ij}(\tilde{m})]_p^\lambda \leq \sum_{p\geq 1} \Lambda_A(\tilde{m}_p)\tilde{m}_p^\lambda$, so that a.s., the map $t \mapsto \sum_{p\geq 1} \Lambda_A(M_p(\tilde{m}, t))M_p(\tilde{m}, t)^\lambda$ is non-increasing (for any $\tilde{m} \in \ell_\lambda$).

Step 3. For $n \geq 1$, denote by $m^n = (m_1, \ldots, m_n, \infty, \ldots)$, and let $k \geq 1$ be fixed. Build the process $Z^n(t)$ as described in Remark 5.1

$$Z^n(t) = Z(0) + \int_0^t \int_{k<l} \int_0^\infty \{\mathbf{c}_{k,l}(Z^n(s-)) - Z^n(s-)\} \\ \times 1_{\{z\leq K[\mu_k(m^n, Z^n(s-)), \mu_l(m^n, Z^n(s-))] f(k,l,Z^n(s-))\}} O(\mathrm{d}s, \mathrm{d}(k,l), \mathrm{d}z), \tag{9.7}$$

with the choice $f(k, l, p) = (m_k^n m_l^n)/(\mu_k(m^n, p)\mu_l(m^n, p))$ and let $X_k(m^n, t) = \mu_k(m^n, Z^n(t))$ be the size of the cluster containing m_k^n at time t. We also put $M(m^n, t) = \mathcal{M}(m^n, Z(t))$. Then one may prove that setting $\varepsilon := m_1 > 0$,

$$\begin{aligned} E[\Lambda_A(X_k(m^n, t))] &\leq \Lambda_A(m_k^n) + \int_0^t \mathrm{d}s E\Bigg[\sum_{p\geq 1} K(X_k(m^n, s-), M_p(m^n, s-)) \\ &\quad \times \{\Lambda_A(X_k(m^n, s-) + M_p(m^n, s-)) - \Lambda_A(X_k(m^n, s-))\}\Bigg] \\ &\leq \Lambda_A(m_k^n) + C_\varepsilon \int_0^t \mathrm{d}s E\Bigg[\sum_{p\geq 1} M_p(m^n, s-)^\lambda \Lambda_A(M_p(m^n, s-))\Bigg]. \end{aligned} \tag{9.8}$$

Using Steps 1 and 2, we obtain

$$E[\Lambda_A(X_k(m^n, t))] \leq \Lambda_A(m_k^n) + C_\varepsilon t \sum_{p\geq 1} (m_p^n)^\lambda \Lambda_A(m_p^n). \tag{9.9}$$

Step 4. We now estimate $M_2(m^n, t)$. To this aim, we note that, M_2 standing for the size of the second smallest particle, $\{X_1(m^n, t) < m_l^n\} \subset \{M_2(m^n, t) \leq X_l(m^n, t)\}$ for any $l \geq 2$. Moreover, due to (9.9)

$$P(X_1(m^n, t) \geq m_l^n) \leq P[\Lambda_A(X_1(m^n, t)) \geq \Lambda_A(m_l^n)] \leq \frac{\Lambda_A(m_1^n) + C_A^n t}{\Lambda_A(m_l^n)} \tag{9.10}$$

with $C_A^n := C_\varepsilon \sum_p (m_p^n)^\lambda \Lambda_A(m_p^n)$. Likewise for any $x > 0$,

$$P(X_l(m^n, t) > x) \leq \frac{\Lambda_A(m_l^n) + C_A^n t}{\Lambda_A(x)}. \tag{9.11}$$

Hence

$$P(M_2(m^n, t) \leq x) \geq P(X_1(m^n, t) < m_l^n; X_l(m^n, t) \leq x)$$

$$\geq 1 - P(X_1(m^n, t) \geq m_l^n) - P(X_l(m^n, t) > x)$$
$$\geq 1 - \frac{\Lambda_A(m_1^n) + C_A^n t}{\Lambda_A(m_l^n)} - \frac{\Lambda_A(m_l^n) + C_A^n t}{\Lambda_A(x)}. \tag{9.12}$$

Now we let n tend to infinity. Since $\lim_n C_A^n = C_A := C_\varepsilon \sum_p m_p^\lambda \Lambda_A(m_p)$, and since $M_2(m^n, t)$ goes in law to $M_2(m, t)$ due to Theorem 2.2,

$$P(M_2(m,t) \leq x) \geq 1 - \frac{\Lambda_A(m_1) + C_A t}{\Lambda_A(m_l)} - \frac{\Lambda_A(m_l) + C_A t}{\Lambda_A(x)}. \tag{9.13}$$

Now we let A tend to infinity, and obtain

$$P(M_2(m,t) \leq x) \geq 1 - \frac{\Lambda(m_1) + Ct}{\Lambda(m_l)} - \frac{\Lambda(m_l) + Ct}{\Lambda(x)}, \tag{9.14}$$

with $C = \lim_{A\to\infty} C_A < \infty$ due to Step 1. Now, for each $l \geq 1$, since $\Lambda(\infty) = \infty$,

$$P(M_2(m,t) < \infty) = \lim_{x\to\infty} P(M_2(m,t) \leq x) \geq 1 - \frac{\Lambda(m_1) + Ct}{\Lambda(m_l)}. \tag{9.15}$$

Letting l grow to infinity, we finally obtain $P(M_2(m,t) < \infty) = 1$.

Step 5. We show that for any $k \geq 3$ and any $t \geq 0$, $M_k(m,t) < \infty$ almost surely. We work by contradiction and assume that for some $k \geq 3$, $t_0 \geq 0$, $P[\Omega_{t_0,k}] > 0$ where $\Omega_{t_0,k} := \{M_k(m, t_0) = \infty\}$. On $\Omega_{t_0,k}$, at time t_0 there exist at most $k-1$ finite particles $M_1(m,t_0), \dots, M_{k-1}(m,t_0)$ that evolve in the system. Since the interaction kernel K is strictly positive on $(0,\infty)^2$, all the particles will have merged together after a finite time: almost surely, on $\Omega_{t_0,k}$, there exists $\tau \in (t_0, \infty)$ such that $M_1(m,\tau)$ contains all the particles. In other words, $M_2(m,s) = \infty$ for all $s \geq \tau$. This implies that there exists $t_1 \geq t_0$ such that $P[M_2(m,t_1) = \infty] \geq P[\Omega_{t_0,k}] > 0$, which contradicts Step 4.

Appendix A. Estimates concerning c_{ij} and d_λ

This section is devoted to fundamental inequalities concerning the action of c_{ij} on d_λ and $\|\,.\,\|_\lambda$. We start with a consequence of [6, Lemma 3.1], which will permit us to neglect as often as possible the reordering after coalescence.

Lemma A.1. *Fix $\lambda \in \mathbb{R} \setminus \{0\}$. Consider any pair of finite permutations $\sigma, \tilde{\sigma}$ of $\mathbb{N}$. Then for all $m, \tilde{m} \in \ell_\lambda$,*

$$d_\lambda(m, \tilde{m}) \leq \sum_{k=1}^{\infty} |m_{\sigma(k)}^\lambda - \tilde{m}_{\tilde{\sigma}(k)}^\lambda|. \tag{A.1}$$

We shall also make use of the following basic inequality: for all $\alpha, \beta > 0$, there exists a constant $C = C_{\alpha,\beta}$ such that for all $x, y \in \mathbb{R}_+$,

$$(x^\alpha + y^\alpha)|x^\beta - y^\beta| \leq 2|x^{\alpha+\beta} - y^{\alpha+\beta}| \leq C(x^\alpha + y^\alpha)|x^\beta - y^\beta|. \tag{A.2}$$

Lemma A.2. *Let $\lambda \in \mathbb{R} \setminus \{0\}$. Assume $A(\lambda)$. There exists a constant C such that, for all $m, \tilde{m} \in \ell_\lambda$, all $1 \leq i < j < \infty$,* *Case* 1*:* $\lambda < 0$,

$$\|c_{ij}(m)\|_\lambda \le \|m\|_\lambda, \tag{A.3}$$

$$d_\lambda(c_{ij}(m), m) \le 2m_i^\lambda, \tag{A.4}$$

$$d_\lambda(c_{ij}(m), c_{ij}(\tilde{m})) \le d_\lambda(m, \tilde{m}). \tag{A.5}$$

Case 2*:* $\lambda \in (0, 1]$,

$$\|c_{ij}(m)\|_\lambda \le \|m\|_\lambda, \tag{A.6}$$

$$d_\lambda(c_{ij}(m), m) \le 2m_j^\lambda, \tag{A.7}$$

$$d_\lambda(c_{ij}(m), c_{ij}(\tilde{m})) \le d_\lambda(m, \tilde{m}). \tag{A.8}$$

Case 3*:* $\lambda > 1$,

$$\|c_{ij}(m)\|_\lambda \le \|m\|_\lambda + 2^{\lambda-1}\lambda m_j m_i^{\lambda-1}, \tag{A.9}$$

$$d_\lambda(c_{ij}(m), m) \le (1 + 2^{\lambda-1})\lambda m_j m_i^{\lambda-1}, \tag{A.10}$$

$$\begin{aligned} &K(m_i, m_j)\left[d_\lambda(c_{ij}(m), c_{ij}(\tilde{m})) - d_\lambda(m, \tilde{m})\right] \\ &\quad \le C\left[(m_i^\lambda + \tilde{m}_i^\lambda)|m_j^\lambda - \tilde{m}_j^\lambda| + (m_j^\lambda + \tilde{m}_j^\lambda)|m_i^\lambda - \tilde{m}_i^\lambda|\right]. \end{aligned} \tag{A.11}$$

Proof. We consider separately the different cases.

Case 1: $\lambda < 0$. First, (A.3) follows from the fact that $(m_i + m_j)^\lambda \le m_i^\lambda + m_j^\lambda$. Let now σ be the finite permutation of $\mathbb{N}$ that achieves

$$c := ((c_{ij}(m))_{\sigma(n)})_{n\ge1} = (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_{j-1}, m_i + m_j, m_{j+1}, \ldots), \tag{A.12}$$

and let $\tilde{c}$, $\tilde{\sigma}$ be the corresponding objects concerning $\tilde{m}$. Then, using Lemma A.1,

$$\begin{aligned} d_\lambda(c_{ij}(m), m) &\le \sum_{k\ge1} |m_k^\lambda - c_k^\lambda| \\ &\le \sum_{k=i}^{j-2} |m_k^\lambda - m_{k+1}^\lambda| + |m_{j-1}^\lambda - (m_i + m_j)^\lambda| + \sum_{k\ge j} |m_k^\lambda - m_{k+1}^\lambda| \\ &= m_i^\lambda - m_{j-1}^\lambda + m_{j-1}^\lambda - (m_i + m_j)^\lambda + m_j^\lambda \le m_i^\lambda + m_j^\lambda \le 2m_i^\lambda, \end{aligned} \tag{A.13}$$

which proves (A.4). We used here that the sequence m_k^λ is non-increasing and tends to 0. Finally, using again Lemma A.1, we easily obtain

$$\begin{aligned} &d_\lambda(c_{ij}(m), c_{ij}(\tilde{m})) \le \sum_{k\ge1} |c_k^\lambda - \tilde{c}_k^\lambda| \\ &\quad = d_\lambda(m, \tilde{m}) + |(m_i + m_j)^\lambda - (\tilde{m}_i + \tilde{m}_j)^\lambda| - |m_i^\lambda - \tilde{m}_i^\lambda| - |m_j^\lambda - \tilde{m}_j^\lambda|. \end{aligned} \tag{A.14}$$

This yields (A.5), since one can check that for all $x, \tilde{x}, y, \tilde{y}$ in $(0, \infty]$, $|(x + y)^\lambda - (\tilde{x} + \tilde{y})^\lambda| - |x^\lambda - \tilde{x}^\lambda| - |y^\lambda - \tilde{y}^\lambda| \le 0$.

Case 2: $\lambda \in (0, 1]$. It was treated in [6, Corollary 3.2].

Case 3: $\lambda \in (1, 2]$. First, (A.9) follows from the inequality (since $m_j \le m_i$)

$$\begin{aligned} (m_i + m_j)^\lambda - m_i^\lambda - m_j^\lambda &\le (m_i + m_j)^\lambda - m_i^\lambda \\ &\le \lambda m_j(m_i + m_j)^{\lambda-1} \le \lambda 2^{\lambda-1} m_j m_i^{\lambda-1}. \end{aligned} \tag{A.15}$$

Denote by σ the finite permutation of $\mathbb{N}$ that achieves

$$c := ((c_{ij}(m))_{\sigma(n)})_{n\geq 1} = (m_1, \ldots, m_{i-1}, m_i + m_j, m_{i+1}, \ldots, m_{j-1}, m_{j+1}, \ldots), \quad \text{(A.16)}$$

and the corresponding $\tilde{\sigma}$, $\tilde{c} = (\tilde{c}_k)_{k\geq 1}$ concerning $\tilde{m}$. Due to Lemma A.1,

$$\begin{aligned} d_\lambda(c_{ij}(m), m) &\leq \sum_{k\geq 1} |c_k^\lambda - m_k^\lambda| \\ &= (m_i + m_j)^\lambda - m_i^\lambda + \sum_{k\geq j} |m_{k+1}^\lambda - m_k^\lambda| \\ &= (m_i + m_j)^\lambda - m_i^\lambda + m_j^\lambda. \end{aligned} \quad \text{(A.17)}$$

We used here the fact that the sequence m_k^λ is non-increasing and tends to 0. Since $m_j^\lambda \leq m_j m_i^{\lambda-1}$, (A.15) allows us to conclude that (A.10) holds. Next, still using Lemma A.1, we get

$$\begin{aligned} d_\lambda(c_{ij}(m), c_{ij}(\tilde{m})) &\leq \sum_{k\geq 1} |c_k - \tilde{c}_k| \\ &\leq d_\lambda(m, \tilde{m}) + |(m_i + m_j)^\lambda - (\tilde{m}_i + \tilde{m}_j)^\lambda| \\ &\quad - |m_i^\lambda - \tilde{m}_i^\lambda| - |m_j^\lambda - \tilde{m}_j^\lambda|. \end{aligned} \quad \text{(A.18)}$$

Tedious computations allow us to get, for all $x, \tilde{x}, y \in [0, \infty)$,

$$|(x+y)^\lambda - (\tilde{x}+y)^\lambda| - |x^\lambda - \tilde{x}^\lambda| \leq \lambda \left[\left(y|x^{\lambda-1} - \tilde{x}^{\lambda-1}| \right) \wedge \left(y^{\lambda-1}|x - \tilde{x}| \right) \right], \quad \text{(A.19)}$$

so that

$$\begin{aligned} &|(m_i + m_j)^\lambda - (\tilde{m}_i + \tilde{m}_j)^\lambda| - |m_i^\lambda - \tilde{m}_i^\lambda| - |m_j^\lambda - \tilde{m}_j^\lambda| \\ &\quad \leq |(m_i + m_j)^\lambda - (\tilde{m}_i + m_j)^\lambda| - |m_i^\lambda - \tilde{m}_i^\lambda| \\ &\qquad + |(\tilde{m}_i + \tilde{m}_j)^\lambda - (\tilde{m}_i + m_j)^\lambda| - |m_j^\lambda - \tilde{m}_j^\lambda| \\ &\quad \leq \lambda m_j |m_i^{\lambda-1} - \tilde{m}_i^{\lambda-1}| + \lambda \tilde{m}_i^{\lambda-1} |m_j - \tilde{m}_j| \\ &\quad \leq C m_j \frac{|m_i^\lambda - \tilde{m}_i^\lambda|}{m_i + \tilde{m}_i} + C \tilde{m}_i^{\lambda-1} \frac{|m_j^\lambda - \tilde{m}_j^\lambda|}{m_j^{\lambda-1} + \tilde{m}_j^{\lambda-1}}. \end{aligned} \quad \text{(A.20)}$$

We used (A.2) to obtain the last inequality. Finally, using (2.8), and the fact that $\lambda/2 \leq 1$, $m_j \leq m_i$, $\tilde{m}_j \leq \tilde{m}_i$, we obtain

$$\begin{aligned} &K(m_i, m_j) \left[d_\lambda(c_{ij}(m), c_{ij}(\tilde{m})) - d_\lambda(m, \tilde{m}) \right] \\ &\quad \leq C (m_i m_j)^{\lambda/2} m_j \frac{|m_i^\lambda - \tilde{m}_i^\lambda|}{m_i + \tilde{m}_i} + C (m_i m_j)^{\lambda/2} \tilde{m}_i^{\lambda-1} \frac{|m_j^\lambda - \tilde{m}_j^\lambda|}{m_j^{\lambda-1} + \tilde{m}_j^{\lambda-1}} \\ &\quad \leq C m_j^\lambda |m_i^\lambda - \tilde{m}_i^\lambda| + C m_i^{\lambda/2} m_j^{1-\lambda/2} \tilde{m}_i^{\lambda-1} |m_j^\lambda - \tilde{m}_j^\lambda|. \end{aligned} \quad \text{(A.21)}$$

The basic inequality $m_i^{\lambda/2} m_j^{1-\lambda/2} \tilde{m}_i^{\lambda-1} \leq m_i \tilde{m}_i^{\lambda-1} \leq m_i^\lambda + \tilde{m}_i^\lambda$ allows us to conclude that (A.11) holds.

Case 4: $\lambda > 2$. First, since (A.15) also holds for $\lambda > 2$, (A.9) and (A.10) can be checked exactly as in Case 3. Next, arguing as in Case 3 again, we obtain that (A.18) still holds. A simple computation shows that for $x, \tilde{x}, y \in [0, \infty)$,

$$
\begin{aligned}
&|(x+y)^\lambda - (\tilde{x}+y)^\lambda| - |x^\lambda - \tilde{x}^\lambda| \le \lambda y |(x+y)^{\lambda-1} - (\tilde{x}+y)^{\lambda-1}| \\
&\quad \le \lambda(\lambda-1) y |x-\tilde{x}| ((x \vee \tilde{x}) + y)^{\lambda-2}.
\end{aligned} \tag{A.22}
$$

Thus, using that $m_j \le m_i$ and $\tilde{m}_j \le \tilde{m}_i$, we obtain

$$
\begin{aligned}
&|(m_i+m_j)^\lambda - (\tilde{m}_i+\tilde{m}_j)^\lambda| - |m_i^\lambda - \tilde{m}_i^\lambda| - |m_j^\lambda - \tilde{m}_j^\lambda| \\
&\quad \le |(m_i+m_j)^\lambda - (\tilde{m}_i+m_j)^\lambda| - |m_i^\lambda - \tilde{m}_i^\lambda| \\
&\qquad + |(\tilde{m}_i+m_j)^\lambda - (\tilde{m}_i+\tilde{m}_j)^\lambda| - |m_j^\lambda - \tilde{m}_j^\lambda| \\
&\quad \le C m_j |m_i - \tilde{m}_i| (m_i^{\lambda-2} + \tilde{m}_i^{\lambda-2}) + C\tilde{m}_i |m_j - \tilde{m}_j| (\tilde{m}_i^{\lambda-2} + m_i^{\lambda-2}) \\
&\quad \le C m_j \frac{|m_i^\lambda - \tilde{m}_i^\lambda|}{m_i^{\lambda-1} + \tilde{m}_i^{\lambda-1}} (m_i^{\lambda-2} + \tilde{m}_i^{\lambda-2}) + C\tilde{m}_i \frac{|m_j^\lambda - \tilde{m}_j^\lambda|}{m_j^{\lambda-1} + \tilde{m}_j^{\lambda-1}} (\tilde{m}_i^{\lambda-2} + m_i^{\lambda-2}),
\end{aligned} \tag{A.23}
$$

where we used (A.2) and (A.22). Using finally (A.18) and (2.10), we get, since $m_j \le m_i$, $\tilde{m}_j \le \tilde{m}_i$,

$$
\begin{aligned}
&K(m_i, m_j) \left[d_\lambda(c_{ij}(m), c_{ij}(\tilde{m})) - d_\lambda(m, \tilde{m}) \right] \\
&\quad \le C m_i m_j^{\lambda-1} \frac{m_j}{m_i^{\lambda-1} + \tilde{m}_i^{\lambda-1}} (m_i^{\lambda-2} + \tilde{m}_i^{\lambda-2}) |m_i^\lambda - \tilde{m}_i^\lambda| \\
&\qquad + C m_i m_j^{\lambda-1} \frac{\tilde{m}_i}{m_j^{\lambda-1} + \tilde{m}_j^{\lambda-1}} (m_i^{\lambda-2} + \tilde{m}_i^{\lambda-2}) |m_j^\lambda - \tilde{m}_j^\lambda| \\
&\quad \le C m_j^\lambda |m_i^\lambda - \tilde{m}_i^\lambda| + C(m_i^\lambda + \tilde{m}_i^\lambda) |m_j^\lambda - \tilde{m}_j^\lambda|.
\end{aligned} \tag{A.24}
$$

This implies (A.11) and thus concludes the proof. □

References

[1] D.J. Aldous, Brownian excursions, critical random graphs and the multiplicative coalescent, Ann. Probab. 25 (2) (1997) 812–854.

[2] D.J. Aldous, J. Pitman, The standard additive coalescent, Ann. Probab. 26 (4) (1998) 1703–1726.

[3] D.J. Aldous, Deterministic and stochastic models for coalescence (aggregation, coagulation): A review of the mean-field theory for probabilists, Bernoulli 5 (1999) 3–48.

[4] R.L. Drake, A general mathematical survey of the coagulation equation, in: Topics in Current Aerosol Research (part 2), International Reviews in Aerosol Physics and Chemistry, Pergamon Press, Oxford, 1972, pp. 203–376.

[5] S.N. Evans, J. Pitman, Construction of Markovian coalescents, Ann. Inst. H. Poincaré Probab. Statist. 34 (3) (1998) 339–383.

[6] N. Fournier, On some stochastic coalescents, Probab. Theory Related Fields 136 (4) (2006) 509–523.

[7] N. Fournier, J.S. Giet, Convergence of the Marcus–Lushnikov process, Methodol. Comput. Appl. Probab. 6 (2) (2004) 219–231.

[8] N. Fournier, Ph. Laurençot, Well-posedness of Smoluchowski's coagulation equation for a class of homogeneous kernels, J. Funct. Anal. 233 (2006) 351–379.

[9] I. Jeon, Existence of gelling solutions for coagulation-fragmentation equations, Comm. Math. Phys. 194 (3) (1998) 541–567.

[10] J.F.C. Kingman, The coalescent, Stochastic Process. Appl. 13 (3) (1982) 235–248.

[11] A. Lushnikov, Some new aspects of coagulation theory, Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosfer. I Okeana 14 (10) (1978) 738–743.

[12] A. Marcus, Stochastic coalescence, Technometrics 10 (1968) 133–143.

[13] J.R. Norris, Smoluchowski's coagulation equation: Uniqueness, nonuniqueness and a hydrodynamic limit for the stochastic coalescent, Ann. Appl. Probab. 9 (1) (1999) 78–109.